STU
SvF
SLOVAK UNIVERSITY OF
TECHNOLOGY IN BRATISLAVA
FACULTY OF CIVIL ENGINEERING

Mgr. Martin Bachratý

Summary of PhD dissertation

Generalised polygons in the degree-diameter problem

Submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics.

Bratislava, 2021

The submitted PhD dissertation has been prepared at the Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava.

Author: Mgr. Martin Bachratý
Department of Mathematics and Descriptive Geometry
Faculty of Civil Engineering, STU, Bratislava

Supervisor: prof. RNDr. Jozef Širáň, DrSc.
Department of Mathematics and Descriptive Geometry
Faculty of Civil Engineering, STU, Bratislava

Referees:

PhD dissertation summary was submitted on 31 May 2021.

PhD dissertation defence will be held online on at am/pm.

prof. Ing. Stanislav Unčík, PhD.
Dean of the Faculty of Civil Engineering

Abstrakt

Najväčší možný rád $n(d,k)$ grafu maximálneho stupňa d a priemeru k nemôže prekročiť moorovskú hranicu $M(d,k)$, ktorá má tvar $M(d,k) = 1+d+d(d-1)+\cdots+d(d-1)^{k-1}$. Hovoríme, že k moorovskej hranici sa dá asymptoticky priblížiť pre priemer k, ak $\limsup_{d\to\infty} n(d,k)/M(d,k) = 1$. Je známe, že to je možné pre priemery 2, 3 a 5. Ako dôkaz slúžia polaritné kvocienty incidenčných grafov konečných zovšeobecnených n-uholníkov majúcich polaritu pre $n \in \{3,4,6\}$.

V tejto práci skúmame možnosti adaptácie týchto výsledkov na cayleyovské grafy veľkého rádu a identického priemeru. Pomocou tohto výskumu sme objavili alternatívnu konštrukciu cayleyovských grafov stupňa d, priemeru 2 a rádu $d^2+o(d^2)$, pôvodne skonštruovaných v roku 2012 Šiagiovou a Širáňom. Okrem toho sme ukázali, že k moorovskej hranici sa dá asymptoticky priblížiť pre priemer 3 cayleyovskými grafmi. Nakoniec sa pozrieme na (ne)použiteľnosť rôznych konštrukcií veľkých cayleyovských grafov pre daný stupeň a priemer 4 alebo 5.

Abstract

The largest order $n(d,k)$ of a graph of maximum degree d and diameter k cannot exceed the Moore bound $M(d,k)$ of the form $M(d,k) = 1+d+d(d-1)+\cdots+d(d-1)^{k-1}$. We say that the Moore bound can be asymptotically approached for diameter k if $\limsup_{d\to\infty} n(d,k)/M(d,k) = 1$. This is known to be true for diameters 2, 3 and 5. It follows by taking polarity quotients of the incidence graphs of finite generalised n-gons admitting a polarity for $n \in \{3,4,6\}$.

In this dissertation we investigate ways of strengthening these findings to large Cayley graphs of the same diameter. As an application we find an alternative construction of the family of Cayley graphs of degree d, diameter 2, and order $d^2+o(d^2)$, first introduced in 2012 by Šiagiová and Širáň. Further, we prove that the Moore bound can be asymptotically approached for diameter 3 by Cayley graphs. Finally, we discuss the (im)possibility of several constructions of large Cayley graphs of given degree and diameter 4 or 5.

Contents

1 Introduction

The core of this dissertation is formed by findings of three research papers [5, 3, 4] in which we examined the possibilities of using the structure of finite generalised polygons to construct large vertex-transitive graphs of small diameter.

The degree-diameter problem, which is the problem of finding the largest order of a graph of given maximum degree and diameter, is one of the classical problems in extremal graph theory. Presumably the earliest significant result in the degree-diameter problem was presented by Hoffman and Singleton in [25], where they determined all possible values of d for which there can exist a certain graph of diameter 2 or 3 and degree d that is maximal according to a certain definition. The problem of finding large graphs of small degree and diameter gained much attention during the last two decades of the 20th century, mainly because because of the obvious applications in network designs. Since then, an extensive research lead to a number of both theoretical and applied results in this area.

The history of finite geometry has roots in the 19th century. One of the first individual examples of a finite geometry dates back to 1849, when Kirkman described what is now known as a $(60_3\, 60_3)$ configuration. The first axiomatic treatment of finite geometries, namely finite projective planes, is due to Fano, who is considered to be the founder of finite geometry. One of the most interesting classes of finite geometries is the class of finite generalised polygons. Generalised polygons, which can be seen as a natural generalisation of projective planes, were introduced by Tits in [43]. These geometries are not only intriguing from a pure geometrical point of view, but they also provide many interesting combinatorial applications.

In 1962 Erdős and Rényi [17] constructed a family of graphs, now also known as polarity graphs, by taking polarity quotients of the incidence graphs of finite projective planes. The role of these graphs in the degree-diameter problem (for diameter 2) was realised several years later, and it quickly lead to an observation by Delorme [15] that similar constructions based on generalised quadrangles and generalised hexagons produce two new infinite families of polarity graphs of large orders and diameters 3 and 5, respectively. Since then, many new techniques for constructing large graphs of given maximum degree and diameter have been developed but, nevertheless, for all but finitely many degrees the largest known graphs of given maximum degree and diameter 2, 3 or 5 are either the polarity graphs or their modifications.

The purpose of this dissertation is to thoroughly explore possible applications of the three aforementioned families of polarity graphs in the degree-diameter problem. Namely, we are interested in how the graphs in these families can be used in the degree-diameter problem for vertex-transitive, and in particular Cayley graphs. Polarity graphs are not regular, so they cannot be vertex-transitive. Nevertheless, since they have a relatively large group of automorphism, it is natural to ask if one can modify them to obtain large vertex-transitive graphs of given degree and small diameter. In Chapter 3 we look closely at symmetry properties of the polarity graphs of finite projective planes. Using these observations, we then develop an alternative method of constructing the largest currently known Cayley graphs of given degree and diameter 2. As we show in Chapter 4, this method can be extended to the polarity graphs of generalised quadrangles, producing new record Cayley graphs for diameter 3. Then, in Chapter 5, we explain why this approach does not carry over to an analogous construction from generalised hexagons, and we also discuss other possible methods of constructing large Cayley graphs of given diameter.

2 Background

In this chapter we present a number of preliminary definitions and properties related to generalised polygons and the degree-diameter problem. All graphs considered in this dissertation are assumed to be finite, simple and connected.

2.1 Preliminaries on generalised polygons

In this section we introduce the notion of a generalised polygon and provide several facts and definitions that will be helpful in later sections and chapters. Most of these are considered folklore, but all can be also found in [46], an excellent introduction to the theory of generalised polygons.

2.1.1 Incidence geometries

Definition 2.1.1. A *geometry* (*of rank* 2)[1] $\mathcal{G}$ is a triple $(\mathcal{P}, \mathcal{L}, \mathbf{I})$, where $\mathcal{P}$ and $\mathcal{L}$ are disjoint non-empty sets and $\mathbf{I} \subseteq \mathcal{P} \times \mathcal{L}$ is a relation.

The elements of $\mathcal{P}$ and $\mathcal{L}$ are the *points* and *lines* of $\mathcal{G}$, respectively, and the relation $\mathbf{I}$ is the *incidence relation* of $\mathcal{G}$. We will sometimes refer to the elements of $\mathcal{P} \cup \mathcal{L}$ as *elements* of $\mathcal{G}$. A geometry is said to be *finite* if $\mathcal{P}$ and $\mathcal{L}$ are finite sets. A geometry is called *thick* if each point lies on at least three lines, and each line passes through at least three points. Throughout this dissertation we will be interested only in finite thick geometries.

If every line of a geometry $\mathcal{G}$ contains the same number of points, say $s+1$, and every point of $\mathcal{G}$ lies on the same number of lines, say $t+1$, we say that $\mathcal{G}$ has *order* (s, t). If $s = t$, then we simply say that $\mathcal{G}$ has order s.

A *subgeometry* of a geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a geometry $\mathcal{G}' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ with $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathbf{I}' = \mathbf{I} \cap (\mathcal{P}' \times \mathcal{L}')$. For each positive integer n, the *ordinary* n*-gon* is the unique connected geometry of order 1 (if $n \geq 2$) or 0 (if $n = 1$) with n points and n lines. Note that each ordinary n-gon arise naturally from a (regular) polygon in the Euclidean space. (For cases $n = 1$ and $n = 2$ we include a monogon and a digon, the two degenerate regular polygons.)

Next we introduce an important graph related to each geometry.

Definition 2.1.2. The *incidence graph* $L(\mathcal{P}, \mathcal{L})$, also called the *Levi graph*, of a geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is defined as the graph having one vertex associated with every element of $\mathcal{P} \cup \mathcal{L}$, and two vertices $a, b \in (\mathcal{P} \cup \mathcal{L})$ adjacent if and only if the elements a and b are incident in $\mathcal{G}$.

[1] Geometries of rank 2 are sometimes called *incidence structures* or *incidence geometries of rank* 2.

Note that the incidence graph of a geometry is always bipartite, with $\mathcal{P}$ and $\mathcal{L}$ being the two parts.

2.1.2 Symmetries of geometries

Let $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a geometry. A *collineation*, or an *automorphism*, of $\mathcal{G}$ is a bijection $\alpha\colon \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ that maps points to points and lines to lines, and preserves the incidence. (That is, for each $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$ we have $p \,\mathbf{I}\, \ell$ if and only if $p^\alpha \,\mathbf{I}\, \ell^\beta$.) A *correlation* of $\mathcal{G}$ is a bijection from $\mathcal{P} \cup \mathcal{L}$ to itself that maps points to lines and lines to points, and preserves the incidence.

The *correlation group* of a geometry $\mathcal{G}$ is the group of all collineations and correlations of $\mathcal{G}$. The set of all collineations of $\mathcal{G}$ form a subgroup of the correlation group of index at most 2. This subgroup is called the *automorphism group* of $\mathcal{G}$, and denoted by $\mathrm{Aut}(\mathcal{G})$.

A *polarity* is a correlation of order 2. A geometry that admits a polarity is said to be *self-polar*. An *absolute point* of a polarity π is a point p such that $p \,\mathbf{I}\, p^\pi$, and an *absolute line* of π is a line ℓ such that $\ell^\pi \,\mathbf{I}\, \ell$.

2.1.3 Generalised polygons

There are several equivalent definitions of a generalised polygon. Here we use the definition by Tits given in [45]. The original definition (also due to Tits) can be found in [43].

Definition 2.1.3 ([46]). Let n be a positive integer. A *generalised n-gon* is a geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ such that:

(i) it has no ordinary k-gon as a subgeometry for $2 \leq k < n$;

(ii) for every pair $x, y \in (\mathcal{P} \cup \mathcal{L})$ there is an ordinary n-gon (as a subgeometry of $\mathcal{G}$) containing both x and y;

(iii) there exists an ordinary $(n+1)$-gon (again as a subgeometry) of $\mathcal{G}$.

It was shown in [21] that finite generalised n-gons exist only if $n = 3$, 4, 6 or 8. We will refer to these generalised polygons as generalised triangles, quadrangles, hexagons and octagons. We also note that all generalised polygons are thick; see [46, Lemma 1.3.2] for example.

The following graph-theoretical characterisation of generalised polygons will be helpful. (See Subsection 2.2.1 for definitions of the diameter and girth of a graph.)

Lemma 2.1.4 ([46]). *A geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a generalised n-gon if and only if the incidence graph of $\mathcal{G}$ is a connected graph of diameter n and girth $2n$, such that each vertex has at least three neighbours.*

2.1.4 Finite generalised polygons admitting a polarity

Throughout this dissertation we will be particularly interested in finite generalised polygons that admit a polarity. The following fact about such geometries will be useful.

Proposition 2.1.5 ([34, 36])**.** *If a finite generalised $2m$-gon of order (s,t) admits a polarity, then $s = t$ and ms is a perfect square.*

Up to this date there are three known infinite families of finite self-polar generalised polygons, which we describe here.

Finite projective planes $\mathrm{PG}(2, q)$

Let q be a prime power, and let $F = \mathrm{GF}(q)$ be the Galois field of order q. The *projective plane* over F, denoted by $\mathrm{PG}(2, q)$, is a geometry whose points are the 1-spaces of F^3, lines are the 2-spaces of F^3, and the incidence is given by symmetrised inclusion. Both points and lines can be represented by *projective triples*, that is, equivalence classes $[\mathbf{a}]$ of non-zero triples $\mathbf{a} = (a_1, a_2, a_3)$ of elements of F, with two triples being equivalent if and only if one is a non-zero multiple of the other. Namely, each point is represented by $[\mathbf{v}]$ where $\mathbf{v}$ is any non-zero vector of the corresponding 1-space, and each line is represented by $[\mathbf{w}]$ where $\mathbf{w}$ is a normal vector of the corresponding 2-space. It is easy to see that a point $[\mathbf{a}]$ and a line $[\mathbf{b}]$ (where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$) are incident if and only if $\mathbf{a}\mathbf{b}^T = a_1b_1 + a_2b_2 + a_3b_3 = 0$. (To make notation clearer we will write $[\mathbf{a}]_{\mathcal{P}}$ if $[\mathbf{a}]$ represents a point and $[\mathbf{a}]_{\mathcal{L}}$ if $[\mathbf{a}]$ represents a line.)

Note that the number of 1-spaces (and consequently also the number of 2-spaces) in F^3 is $q^2 + q + 1$. Also every 1-space is contained in exactly $q + 1$ two-spaces, and every 2-space contains exactly $q+1$ one-spaces. It follows that $\mathrm{PG}(2, q)$ has q^2+q+1 points, q^2+q+1 lines, and order q. The following observation is an easy exercise (and a well known fact).

Lemma 2.1.6. *The projective plane over any finite field is a finite generalised triangle.*

Let $\mathrm{PG}(2, q) = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be the projective plane over $\mathrm{GF}(q)$. The mapping $\pi\colon \mathcal{P}\cup\mathcal{L} \to \mathcal{P}\cup\mathcal{L}$ that interchanges $[\mathbf{a}]_{\mathcal{P}}$ and $[\mathbf{a}]_{\mathcal{L}}$ (for each projective triple $[\mathbf{a}]$) is called the *standard polarity* of $\mathrm{PG}(2, q)$. The order of π is clearly two, and since π also interchanges points and lines of $\mathrm{PG}(2, q)$ and preserves the incidence, we find that π is a polarity of $\mathrm{PG}(2, q)$. It follows that every projective plane over a finite field is self-polar. For further information on polarities of finite projective planes we refer the reader to [6, 22].

Symplectic quadrangles $\mathrm{W}(2^{2e+1})$

Let q be a prime power, let $F = \mathrm{GF}(q)$ be the Galois field of order q, and let $\mathrm{PG}(3, q)$ be the projective geometry over F. The points of $\mathrm{PG}(3, q)$ (that is, the 1-spaces of F^4) may be represented by equivalence classes $[\mathbf{a}]$ of non-zero quadruples $\mathbf{a} = (a_1, a_2, a_3, a_4)$ of elements of F, with the same equivalence as for projective triples. Next, let $Q\colon F^4 \times F^4 \to F$ be a skew-symmetric bilinear form. This form is usually taken to be $(\mathbf{a}, \mathbf{b}) \mapsto a_0b_1 - a_1b_0 + a_2b_3 - a_3b_2$ but here we prefer to use an equivalent form given by $Q(\mathbf{a}, \mathbf{b}) = a_0b_3 - a_3b_0 + a_1b_2 - a_2b_1$ for every $\mathbf{a}, \mathbf{b} \in F^4$. A line of $\mathrm{PG}(3, q)$ is said to be *totally isotropic* (with respect to Q) if for any two vectors $\mathbf{v}$ and $\mathbf{w}$ of the corresponding 2-space we have $Q(\mathbf{v}, \mathbf{w}) = 0$. The *symplectic quadrangle* $\mathrm{W}(q)$ is a geometry whose points are the points of $\mathrm{PG}(3, q)$, lines are the totally isotropic line of $\mathrm{PG}(3, q)$, and the incidence is given by symmetrised inclusion.

Similar to the previous case of finite projective planes it is easy to see that the order of $\mathrm{W}(q)$ is q. The number of points of $\mathrm{W}(q)$ is equal to the number of 1-spaces in $\mathrm{PG}(3, q)$, which is q^3+q^2+q+1. It requires a little work to determine the number of lines of $\mathrm{W}(q)$. First we note that for every point $[\mathbf{a}]$ there are q^3 vectors $\mathbf{b}$ such that $Q(\mathbf{a}, \mathbf{b}) = 0$. Since q of those vectors

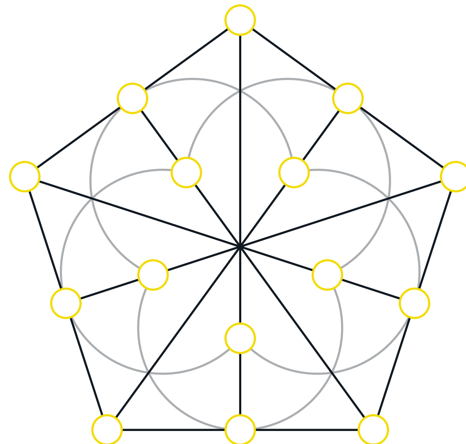


Figure 2.1: The Doily

are scalar multiples of $\mathbf{a}$, we find that there are exactly $(q^3 - q)/(q - 1)$ points $[\mathbf{b}]$ such that the line of $\mathrm{PG}(3, q)$ given by $\mathbf{a}$ and $\mathbf{b}$ is totally isotropic. On the other hand, there are $q(q+1)$ ways how to chose an ordered pair of two distinct points on a line of $\mathrm{W}(q)$, and by an easy counting argument it follows that $\mathrm{W}(q)$ has $q^3 + q^2 + q + 1$ lines. The symplectic quadrangle $\mathrm{W}(2)$ (sometimes referred to as the Doily) shown in Figure 2.1 is the smallest example of a generalised quadrangle. (The lines of $\mathrm{W}(2)$ are represented by 10 line segments and 5 circle arcs.)

It is a well-known fact that all symplectic quadrangles are examples of generalised quadrangles. By Proposition 2.1.5 we know that not every symplectic quadrangle is self-polar. In particular, if $\mathrm{W}(q)$ admits a polarity, then q must be an odd power of two. A classical result of Tits [44] shows that the "if" in this statement can be replaced by "if and only if". We will describe a polarity of $\mathrm{W}(q)$ for each such q next, following [37] but using the skew-symmetric bilinear form Q introduced earlier.

Let $q = 2^{2e+1}$ for some non-negative integer e, let $F = \mathrm{GF}(q)$, let $\omega = 2^{e+1}$, and let σ be the automorphism of F given by $\sigma(x) = x^\omega$ for each $x \in F$. (Note that $\sigma^2(x) = x^2$.) Also for every $\mathbf{a} \in F^4$ we let $c_{\mathbf{a}} = a_0 a_3 + a_1 a_2$, and for every pair of vectors $\mathbf{a}, \mathbf{b} \in \mathrm{PG}(3, q)$ we let $\delta_{ij} = a_i b_j + a_j b_i$ for any $i, j \in \{0, 1, 2, 3\}$. (Dependence of δ_{ij} on $\mathbf{a}$ and $\mathbf{b}$ will always be assumed but not shown in the notation.) A polarity π of $\mathrm{W}(q)$ can now be defined as follows:

Proposition 2.1.7 ([4, Proposition 2.1])**.** *For a point $p = [\mathbf{a}]$ of $\mathrm{W}(q)$ let p^π be the set of all non-zero vectors of F^4 spanned by the four vectors*

$$(0, a_0^\omega, a_1^\omega, c_{\mathbf{a}}^{\omega/2}),\ (a_0^\omega, 0, c_{\mathbf{a}}^{\omega/2}, a_2^\omega),\ (a_1^\omega, c_{\mathbf{a}}^{\omega/2}, 0, a_3^\omega),\ \text{and } (c_{\mathbf{a}}^{\omega/2}, a_2^\omega, a_3^\omega, 0).$$

Conversely, for a line ℓ of $\mathrm{W}(q)$ through a pair of distinct points $[\mathbf{a}]$ and $[\mathbf{b}]$ let ℓ^π be the point $[\mathbf{d}]$ of $\mathrm{W}(q)$ with coordinates given by

$$d_0 = \delta_{01}^{\omega/2},\ d_1 = \delta_{02}^{\omega/2},\ d_2 = \delta_{13}^{\omega/2},\ \text{and } d_3 = \delta_{23}^{\omega/2}.$$

Then p^π is a line of $\mathrm{W}(q)$, and the mapping π is a polarity of $\mathrm{W}(q)$.

Split Cayley hexagons $\mathrm{H}(3^{2e+1})$

There are several ways to define split Cayley hexagons. A construction from a triality of a nondegenerate hyperbolic quadrin in $\mathrm{PG}(7, q)$ can be found in [43], while an alternative method of construction using a split Cayley algebra is given in [39] and [40]. Here we describe split Cayley hexagons using the coordinatisation theory as in [46, Section 3.5].

Let q be a prime power, and let $F = \mathrm{GF}(q)$ be the Galois field of order q. Both points and lines of the split Cayley hexagon $\mathrm{H}(q)$ will be represented by the ordered i-tuples of elements of F for $i \in \{0, 1, 2, 3, 4, 5\}$. In order to distinguish between points and lines, the coordinates will be written in parentheses and square brackets, respectively. Also, for clarity, we will write (∞) and $[\infty]$ for the 0-tuples $()$ and $[]$. Next we describe the incidence relation. The point (∞) (or the line $[\infty]$) is incident with $[\infty]$ (or (∞)) and all the lines (or points) represented by a 1-tuple. Next, for each element x of $\mathrm{H}(q)$ represented by an i-tuple with $i \in \{1, 2, 3, 4\}$, the elements incident with x are obtained by either removing the last coordinate or adding an extra coordinate of any value to the end, and then changing the parentheses with the square brackets, or vice versa. For example, the points incident with the line $[k, b]$ are precisely the points (k, b, k'), where k' ranges over all of F, and the point (k). Finally, a point $(a, \ell, a', \ell', a'')$ and a line $[k, b, k', b', k'']$ are incident if and only if

$$\begin{aligned} b &= -ak + a'', \\ k' &= a^3k^2 + \ell' - \ell k - 3a^2a''k - 3a'a'' + 3aa''^2, \\ b' &= a^2k + a' - 2aa'', \\ k'' &= a^3k + \ell - 3a''a^2 + 3aa'. \end{aligned} \tag{2.1}$$

It is easy to see that $\mathrm{H}(q)$ has $q^5 + q^4 + q^3 + q^2 + q + 1$ points, $q^5 + q^4 + q^3 + q^2 + q + 1$ lines, and with some work it can be shown that the order of $\mathrm{H}(q)$ is q. It can also be proved that $\mathrm{H}(q)$ is self-polar if and only if $q = 3^{2e+1}$ for some non-negative integer e. The "only if" part of the statement follows by Proposition 2.1.5. The truth of the "if" part can be shown by an explicit construction of a polarity. Let θ be the *Tits automorphism* of $\mathrm{GF}(3^{2e+1})$, that is, the automorphism of $\mathrm{GF}(3^{2e+1})$ given by $x \mapsto x^{3^{e+1}}$. Then the involutory mapping ρ that interchanges $(a, \ell, a', \ell', a'')$ and $[a^\theta, \ell^{\theta^{-1}}, a'^\theta, \ell'^{\theta^{-1}}, a''^\theta]$ is a polarity of $\mathrm{H}(3^{2e+1})$. (The action of ρ on i-tuples for $0 \le i \le 4$ is given by the restriction to the first i coordinates of a 5-tuple.)

Note that (2.1) contains many terms with the coefficient 3. For this reason the equations become significantly simpler when $\mathrm{GF}(q)$ has characteristic three. (As is the case with all self-polar split Cayley hexagons.) Namely, if $q = 3^h$, then a point $(a, \ell, a', \ell', a'')$ and a line $[k, b, k', b', k'']$ are incident if and only if

$$\begin{aligned} b &= -ak + a'', \\ k' &= a^3k^2 + \ell' - \ell k, \\ b' &= a^2k + a' - 2aa'', \\ k'' &= a^3k + \ell. \end{aligned} \tag{2.2}$$

Further information on the coordinatisation of split Cayley hexagon (including proofs that $\mathrm{H}(q)$ is a finite generalised hexagons, and that ρ is a polarity) can be found in [14].

2.1.5 Polarity graphs of generalised polygons

We now introduce a family of graphs related to self-polar geometries that will play a key role in later chapters.

Definition 2.1.8. Let $\mathcal{G}$ be a self-polar geometry and let π be a polarity of $\mathcal{G}$. The *polarity graph* $P_{\mathcal{G},\pi}$ is a graph whose vertices are the points of $\mathcal{G}$, with two distinct vertices p_1 and p_2 adjacent if and only if the point p_1 lies on the line p_2^π.

First we note that by the definition of a polarity a point p_1 lies on a line p_2^π if and only if p_2 lies on p_1^π, and hence $P_{\mathcal{G},\pi}$ is a well-defined simple graph. Also note that a polarity graph $P_{\mathcal{G},\pi}$ can be defined equivalently as the quotient graph of the incidence graph of $\mathcal{G}$ obtained by the factorisation by π, that is, by identifying p with p^π throughout and suppressing eventual edges between p and p^π.

In what follows we let $B(q)$ denote the polarity graphs $P_{\mathrm{PG}(2,q),\pi}$, where π is the standard polarity of the corresponding projective plane, $A(q)$ the polarity graphs $P_{\mathrm{W}(q),\pi}$, where $q = 2^{2e+1}$ and π is given by Proposition 2.1.7, and $I(q)$ the polarity graphs $P_{\mathrm{H}(q),\rho}$, where $q = 3^{2e+1}$ and ρ is defined as in Subsection 2.1.4. Whenever we use notation $B(q)$, $A(q)$ or $I(q)$, we assume an appropriate value of q. Some useful properties of polarity graphs $B(q)$, $A(q)$ and $I(q)$ are given in the following observation.

Proposition 2.1.9. *Graphs $B(q)$, $A(q)$ and $I(q)$ have orders q^2+q+1, q^3+q^2+q+1 and $q^5+q^4+q^3+q^2+q+1$, respectively, and maximum degree $q+1$. Moreover, all vertices of smaller degree have degree q, and there are exactly $q+1$, q^2+1 and q^3+1 of them.*

Proof. The first part follows by Definition 2.1.8 and the properties of generalised polygons given in Section 2.1.4. To prove the second part, let Γ be one of the graphs $B(q)$, $A(q)$ or $I(q)$, and let π be a polarity of a geometry $\mathcal{G}$ such that $\Gamma = P_{\mathcal{G},\pi}$. Then (again by Definition 2.1.8) a vertex p of Γ has degree q if and only if p is an absolute point of π. If $\Gamma = B(q)$ and $p = [\mathbf{a}]$, then this is equivalent to $a_1^2 + a_2^2 + a_3^2 = 0$. It is an easy exercise (see [5, Lemma 2.1] for example) to show that this equation has exactly q^2-1 non-zero solutions, giving us a total of $q+1$ absolute points. Similarly, the number of absolute points with respect to any polarity of $\mathrm{W}(q)$ or $\mathrm{H}(q)$ is q^2+1 or q^3+1, respectively (see [46, Proposition 7.2.3]), and the rest follows easily. □

2.2 The degree-diameter problem

In this section, we first give some background from graph theory, and then we introduce some of the most important questions related to the degree-diameter problem. In particular, we look in detail at the degree-diameter problem for vertex-transitive and Cayley graphs.

2.2.1 Background on graph theory

Let Γ be a graph. We will use $V(\Gamma)$ and $E(\Gamma)$ to denote the vertex set and the edges set of Γ. Given any two vertices $u, v \in V(\Gamma)$, a *uv-path* is a path between u and v. The *distance* $d(u,v)$ of two vertices $u, v \in V(\Gamma)$ (in Γ) is the length of a shortest uv-path. The *diameter* of a graph Γ, usually denoted by k, is the largest distance between any pair of vertices of Γ. A graph Γ is *regular* if all vertices of Γ have the same degree. If Γ is regular and all vertices have degree d, we sometimes say that Γ is *d-regular*, or that Γ is *of degree d*. The *girth* of a graph is the length of its shortest cycle. (If a graph does not contain any cycles, its girth is defined to be infinity.) Given a subset V' of $V(\Gamma)$, the subgraph of Γ *induced* by V' is a graph Γ' with $V(\Gamma') = V'$ and $E(\Gamma') \subseteq E(\Gamma) \cap (V' \times V')$. Finally, we say that a subgraph Γ' of Γ is *spanning* if $V(\Gamma') = V(\Gamma)$.

If Γ and Γ' are two graphs, then an *isomorphism* from Γ to Γ' is a bijection from $V(\Gamma)$ to $V(\Gamma')$ that preserves the adjacency of vertices. An *automorphism* of a graph Γ is an isomorphism from Γ to itself. All automorphism of Γ form the *automorphism group of* Γ, denoted by $\mathrm{Aut}(\Gamma)$.

Recall that an action of a group G on a set X is *transitive* if for every $x, y \in X$ there exists $g \in G$ such that $x^g = y$. If for every $x, y \in X$ there exists exactly one $g \in G$ such that $x^g = y$, then the action is said to be *regular*. A graph Γ is *vertex-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the vertices of Γ. An important class of vertex-transitive graphs are Cayley graphs, which we define next.

Definition 2.2.1. Let G be a group, and let X be a generating set for G, such that X is closed under taking inverses, and $1_G \notin X$. The Cayley graph $C(G, X)$ is a graph whose vertex set consist of all elements of G, with two distinct vertices $g, h \in G$ adjacent if and only if $h^{-1}g \in X$.

The two conditions on X imply that $C(G, X)$ is a well-defined simple graph, and since X generates G, we also see that $C(G, X)$ is connected. A graph that is (isomorphic to) a Cayley graph for a cyclic group is called a circulant graph.

The following classical theorem attributed to Sabidussi (see [38]) provides a useful characterisation of Cayley graphs.

Theorem 2.2.2 ([38])**.** *A graph Γ is a Cayley graph for some group G if and only if* $\mathrm{Aut}(\Gamma)$ *has a subgroup isomorphic to G acting regularly on $V(\Gamma)$.*

An immediate consequence of Theorem 2.2.2 is that every Cayley graph is vertex-transitive.

2.2.2 Graphs of given maximum degree and diameter

For positive integers d and k let $n(d, k)$ denote the largest order of a graph of maximum degree d and diameter k. The degree-diameter problem is the problem of finding the number $n(d, k)$. The answer is easy for the cases where $d \leq 2$ or $k = 1$. The only (connected) graphs of maximum degree 1 are the complete graphs K_1 and K_2, and hence $n(1, k)$ is defined only for $k = 1$ and $n(1, 1) = 2$. The largest graph of maximum degree 2 and diameter k is the cycle graph with $2k + 1$ vertices, and hence $n(2, k) = 2k + 1$. Finally, the largest graph of maximum degree d and diameter 1 is the complete graph K_{d+1}, and it follows that $n(d, 1) = d + 1$.

Setting trivial cases aside, the problem of determining the number $n(d, k)$ turns out to be very difficult. In fact, up to this date $n(d, k)$ is known only for seven pairs (d, k), namely $(d, 2)$ with $3 \leq d \leq 7$ and $(3, k)$ with $2 \leq k \leq 4$.

Let Γ be a graph of maximum degree d and diameter k, and take any $v \in V(\Gamma)$. Since each vertex u at distance i from v can have at most $d - 1$ neighbours at distance $i + 1$ from v (unless $i = 0$, in which case $u = v$ and the number of neighbours at distance 1 is at most d), it follows that for each $j \geq 1$ there are at most $d(d-1)^{j-1}$ vertices at distance j from v. Since Γ has diameter k, there are no vertices at distance $k + 1$ or larger from v, and hence

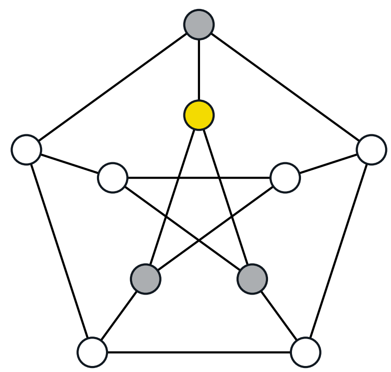


Figure 2.2: Petersen graph with a root vertex

$|V(\Gamma)| \leq 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}$. This upper bound on the order of a graph of maximum degree d and diameter k is known as the Moore bound, and denoted by $M(d,k)$.

Example 2.2.3. Figure 2.2 shows the Petersen graph, the smallest vertex-transitive non-Cayley graph. It is easy to see that the grey vertices are at distance 1 from the yellow vertex, and the white vertices are at distance 2 from the yellow vertex. Then by vertex-transitivity we find that the diameter of the Petersen graph is 2. Also, since the Petersen graph is 3-regular and has 10 vertices, we deduce that $10 \leq n(3,2)$. On the other hand, we have $n(3,2) \leq M(3,2) = 10$, and hence $n(3,2) = 10$.

As shown in Example 2.2.3, the order of the Petersen graph is equal to the Moore bound $M(d,k)$ for the corresponding values $d = 3$ and $k = 2$. Graphs with this property, that is, graphs of degree d, diameter k, and order $M(d,k)$ are called *Moore graphs*. The complete graphs K_n for $n \geq 2$ and the cycle graphs with $2m+1$ vertices are the examples of Moore graphs of orders $M(n-1,1)$ and $M(2,m)$, respectively. If $d \geq 3$ and $k \geq 2$, then $n(d,k) = M(d,k)$ only for $k = 2$ and $d = 3$, 7, and possibly 57, and the unique Moore graphs for the first two degrees are the Petersen graph and the Hoffman-Singleton graph; see [25, 7, 13]. Similar to the Petersen graph, the Hoffman-Singleton graph is also a vertex-transitive non-Cayley graph. The existence of a Moore graph Γ of degree 57, diameter 2, and order 3250 is still an open problem, but if it exists, then $|\mathrm{Aut}(\Gamma)| \leq 375$ (and if $|\mathrm{Aut}(\Gamma)|$ is even, then $|\mathrm{Aut}(\Gamma)| \leq 110$) and, in particular, Γ is not vertex-transitive; see [32]. For further information on the 'missing' Moore graph we refer to a recent survey-type paper [12].

One of the most studied problems related to the Moore bound is the question posed in [8], as to whether for every positive integer c there exists d and k such that $n(d,k) < M(d,k) - c$. A substantial progress in this problem has been made in [19] by showing that for any fixed d and any positive integer c the largest order of a vertex-transitive graph of degree d and diameter k is smaller than $M(d,k) - c$ for almost all k. Another well-known problem in this field is the problem formulated by Delorme in [15], which we explain in the following section.

2.2.3 Asymptomatically approaching the Moore bound

The following problem appears to be one of the driving forces in the degree-diameter problem.

Definition 2.2.4. *Delorme's problem* is the problem of determining $\limsup_{d\to\infty} n(d,k)/M(d,k)$ for every fixed $k \geq 2$.

Note that the Moore bound has the form $M(d,k) = d^k + o(d^k)$, and hence for each k we have $\limsup_{d\to\infty} n(d,k)/M(d,k) = \limsup_{d\to\infty} n(d,k)/d^k$. Also, since for every pair d,k we have $n(d,k) \leq M(d,k)$, it follows that $\limsup_{d\to\infty} n(d,k)/M(d,k) \leq 1$.

We now look at Delorme's problem for diameters 2, 3 and 5. By Proposition 2.1.9 we find that polarity graphs $B(q)$, $A(q)$ and $I(q)$ have maximum degree $q+1$. Next, by Lemma 2.1.4 we know that the diameter of the incidence graph of a generalised n-gon is n. It is clear that the diameter of a polarity quotient of the incidence graph must be less than diameter n of the incidence graph, and hence $B(q)$, $A(q)$ and $I(q)$ have diameters 2, 3 and 5, respectively. Since $B(q)$, $A(q)$ and $I(q)$ exist for an infinite set of degrees $d = q+1$, and their orders have the form $d^2 + o(d^2)$, $d^3 + o(d^3)$ and $d^5 + o(d^5)$, it follows that $\limsup_{d\to\infty} n(d,k)/M(d,k) = 1$ for $k \in \{2,3,5\}$. (This was first observed in [15].)

For the remaining diameters the best available results are much weaker, but far from easy to prove. It was shown in [16] that $\limsup_{d\to\infty} n(d,4)/M(d,4) \geq 1/4$, and for $k \geq 6$ we have $\limsup_{d\to\infty} n(d,k)/M(d,k) \geq (1.6)^{-k}$, where 1.6 can be replaced by 1.57 for $k \equiv -1, 0, 1 \pmod 6$; see [10].

There are various methods for constructing graphs of relatively small maximum degree d and diameter k, and many of the largest currently known graphs for these values are not vertex-transitive. But in the cases when d and k exceeds values manageable by these methods, computer generation of large graphs of maximum degree d and diameter k is almost exclusively limited to searching over Cayley graphs; see [28] and [11]. For this reason it is natural to ask for which diameters the Moore bound can be asymptotically approached (in the sense of Delorme's limit superior being equal to 1) by Cayley graphs. We formalise this in the following definition.

Definition 2.2.5. Let $Cay(d,k)$ denote the largest order of a Cayley graph of degree d and diameter k, and let $Cay(k) = \limsup_{d\to\infty} Cay(d,k)/M(d,k)$. We say that the Moore bound can be *asymptotically approached* for diameter k by Cayley graphs if $Cay(k) = 1$.

In most of the cases, the best available estimates for $Cay(k)$ are not as good as those for $\limsup_{d\to\infty} n(d,k)/M(d,k)$. For each $k \geq 3$ constructions of Cayley graphs in [30, 31] give $Cay(k) \geq k \cdot 3^{-k}$, and the lower bounds can be improved by [47] to $3 \cdot 2^{-4}$, $32 \cdot 5^{-4}$ and $25 \cdot 4^{-5}$ for $k = 3$, 4 and 5, respectively. The strongest finding so far in this area is the fact that $Cay(2) = 1$, showing that the Moore bound for diameter 2 can be asymptotically approached by Cayley graphs. This was shown in [41] by a direct construction of Cayley graphs of degree d, diameter 2, and order $d^2 - O(d^{3/2})$ for an infinite set of degrees d. We will frequently refer to this construction, so we reproduce it for the reader's convenience later in this subsection. As we will see in Chapter 4, this construction is equivalent to extending a regular orbit of a graph $B(q)$ for even prime power q under the action of a suitable group. We will also show by using a variant of this geometric method that for an infinite set of values d there exists a Cayley graph of degree d, diameter 3, and order $d^3 - O(d^{2.5})$. In particular, this proves that the Moore bound can be asymptotically approached for diameter 3 by Cayley graphs. For $k \in \{4,5,6\}$ the bounds for $Cay(k)$ can be improved to 2^{-k} by taking Cartesian products of the record-holding Cayley graphs (of suitable degrees) for diameters 2 and 3. (This follows by the fact that the Cartesian product of two d-regular Cayley graphs of diameters k_1 and k_2, and orders n_1 and n_2, is a Cayley graph of degree $2d$, diameter k_1+k_2, and order n_1n_2.) In the case of vertex-transitive graphs the record-holders for $k \geq 7$ are the graphs of Faber-Moore-Chen type, obtained from the digraphs of Faber, Moore and Chen [20] by ignoring directions and suppressing potential parallel edges.

The degree-diameter problem is often considered for Cayley graphs of abelian, or even cyclic groups; see [48] and [9] for example. Even though we are interested only in the degree-diameter problem for Cayley graphs of general groups, the following observation (see [18] for a short proof) about circulants will be helpful.

Proposition 2.2.6. *Let $n \geq 3$, and let $\mathbb{Z}_n$ be a cyclic subgroup of order n. Then there exists a circulant for $\mathbb{Z}_n$ of diameter 2 and maximum degree at most $2\lceil\sqrt{n}\,\rceil$.*

We now describe the construction of large Cayley graphs of diameter 2 presented in [41].

Construction 2.2.7. Let $e \geq 2$ be an integer, let $F = \mathrm{GF}(2^e)$ be the Galois field of order $q = 2^e$, and let G be the one-dimensional affine group over F represented as the semidirect product $G = F^+ \rtimes F^*$ with the multiplication given by $(a,b)(c,d) = (a+bc, bd)$. Note that F^+ is isomorphic to the elementary abelian group of order 2^e, and hence the elements of F^+ can be represented by vectors of the e-dimensional vector space over GF(2). Let A_1 and A_2 be the set of all non-zero vectors with the first $\lfloor e/2 \rfloor$ coordinates and the last $\lceil e/2 \rceil$ coordinates being equal to 0, respectively, and define $A = \{(a,1) \in G \mid a \in A_1 \cup A_2\}$. Note that every element in G of the form $(z,1)$ is a product of at most two element in A, and that A is closed under taking inverses.

Next, recall that the multiplicative group F^* of a finite field F is always cyclic, and hence by Proposition 2.2.6 there exists a subset B' of F^* such that $|B| \leq 2\left\lceil\sqrt{q-1}\right\rceil$ and $C(F^*, B')$ has diameter two. It follows that every element in G of the form $(0,z)$ with $z \neq 1$ is a product of at most two element in $B = \{(0,b) \in G \mid b \in B'\}$ and, moreover, B is closed under taking inverses.

Finally, let $C = \{(c,c^2) \in G \mid c \in F \text{ and } c \neq 0\}$. Noting that $(c,c^2)(c^{-1},c^{-2}) = (0,1)$, we find that C is closed under taking inverses. Now let (r,s) be any element of G such that $r \neq 0$ and $s \neq 1$. We will show that $(r,s) = (x,x^2)(y,y^2)$ for some pair of non-zero elements $x, y \in F$, by finding a solution of the following equations:

$$r = x + x^2 y, \tag{2.3}$$
$$s = x^2 y^2. \tag{2.4}$$

Since F has characteristic 2 and $s \neq 1$, there exists a unique $t \in F^*$ such that $t \neq 1$ and $s = t^2$. It follows that $xy = t$, and consequently $r = x(1+t)$. But $t \neq 1$, and hence $1+t$ is invertible in F, and we find that $x = r(1+t)^{-1}$ and $y = t(1+t)r^{-1}$.

We have shown that every element in G is a product of at most two elements in $A \cup B \cup C$, and since this set is closed under taking inverses and does not contain the identity element, it follows that $C(G, A \cup B \cup C)$ is a Cayley graph of degree $q + O(\sqrt{q})$, diameter 2, and order $q(q-1)$. Letting $d = q + O(\sqrt{q})$, we find that for an infinite set of values d there exists a Cayley graph of degree d, diameter 2, and order $d^2 + o(d^2)$.

For further information on the degree-diameter problem in the context of how closely can the Moore bound be approached by general, vertex-transitive, or Cayley graphs we recommend [33].

3 Polarity graphs and their symmetries

In this chapter, we first determine the automorphism groups of polarity graphs $B(q)$, $A(q)$ and $I(q)$, and then we derive various properties of $B(q)$ (mostly related to the action of its automorphism group on vertices). In particular, we show that for any sufficiently large odd prime power q there is no vertex transitive graph of degree at most $q+3$ which contains $B(q)$ as a spanning subgraph, and also that for each $e \geq 2$ the only non-edgeless graph induced by an orbit of $\mathrm{Aut}(B(2^e))$ on the vertices of $B(2^e)$ is a vertex-transitive non-Cayley graph.

3.1 Automorphism groups of polarity graphs

First, we give some background in group theory. The 3-*dimensional projective orthogonal group* $\mathrm{PGO}(3,q)$ is the factor group of the subgroup of $\mathrm{GL}(3,q)$ consisting of orthogonal matrices by the centre of this group. The group $\mathrm{P\Gamma O}(3,q)$ is the obvious extension of $\mathrm{PGO}(3,q)$ by Galois automorphisms of $\mathrm{GF}(q)$. The *Suzuki group* $\mathrm{Sz}(2^{2e+1})$ is the subgroup of $\mathrm{Aut}(\mathrm{W}(2^{2e+1}))$ stabilising the set of absolute elements of the polarity π of $\mathrm{W}(2^{2e+1})$ defined in Proposition 2.1.7. The *Ree group* $\mathrm{Ree}(3^{2e+1})$[1] is the subgroup of $\mathrm{Aut}(\mathrm{H}(3^{2e+1}))$ stabilising the set of absolute elements of the polarity ρ of $\mathrm{H}(3^{2e+1})$ described in Subsection 2.1.4. Both Suzuki and Ree groups form an infinite family of groups of Lie type, and they are all simple except for $\mathrm{Sz}(2)$ and $\mathrm{Ree}(3)$; see [44] and [46, Section 7.7].

The automorphism group of $B(q)$ was determined in [35]. In [5] we provided a different and shorter proof, which includes a more detailed discussion on groups. In the following theorem we determine the automorphism groups also for $A(q)$ and $I(q)$.

Theorem 3.1.1. *The automorphism groups of $B(q)$, $A(q)$ and $I(q)$ appears in Table 3.1*

Proof. For $B(q)$ this is [5, Theorem 3.1]. We will show that $\mathrm{Aut}(A(q)) \cong \mathrm{Sz}(q)$. First, let α be an automorphism of $A(q)$, let $\mathcal{P}$ and $\mathcal{L}$ denote the points and lines of the corresponding symplectic quadrangle $\mathrm{W}(q)$, and let π be the polarity of $\mathrm{W}(q)$ such that $A(q) = P_{\mathrm{W}(q),\pi}$. Also let $\overline{\alpha}\colon \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ be a bijection given by $p^{\overline{\alpha}} = p^{\alpha}$ and $\ell^{\overline{\alpha}} = \ell^{\pi\alpha\pi}$ for every $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$. By Definition 2.1.8 we know that p and ℓ are incident in $\mathrm{W}(q)$ if and only if p and ℓ^{π} are adjacent in $A(q)$, which is true exactly when p^{α} and $\ell^{\pi\alpha}$ are adjacent in $A(q)$. But this is equivalent to p^{α} and $\ell^{\pi\alpha\pi}$ being incident in $\mathrm{W}(q)$, and it follows that $\overline{\alpha}$ is an automorphism of $\mathrm{W}(q)$. Moreover, since the absolute points (of π) are exactly the vertices of $A(q)$ of degree q, it follows that $\overline{\alpha}$ preserves the set of absolute elements.

Next, let σ be an automorphism of $\mathrm{W}(q)$ stabilising the set of absolute elements for π. Then the polarity $\sigma^{-1}\pi\sigma$ has the same set of absolute elements as π. But two polarities of $\mathrm{W}(q)$ with

[1] In some literature Ree groups $\mathrm{Ree}(q)$ are denoted by ${}^2G_2(q)$.

Graph	Automorphism group	Comments	Order
$B(q)$	$\mathrm{P\Gamma O}(3,q)$	$q = p^n$ for p prime	$nq(q^2-1)$
$A(q)$	$\mathrm{Sz}(q)$	$q = 2^{2e+1}$ for $e \geq 0$	$q^2(q^2+1)(q-1)$
$I(q)$	$\mathrm{Ree}(q)$	$q = 3^{2e+1}$ for $e \geq 0$	$q^3(q^3+1)(q-1)$

Table 3.1: Automorphism groups of polarity graphs

the same sets of absolute elements coincide (see [46, Corollary 7.6.3]), and hence σ centralises π. In particular, we have $p^{\sigma\pi} = p^{\pi\sigma}$ for every $p \in \mathcal{P}$. Two vertices p_1 and p_2 of $A(q)$ are adjacent if and only if p_1 and p_2^π are incident in $\mathrm{W}(q)$. Since σ is an automorphism of $\mathrm{W}(q)$, this is true exactly when p_1^σ and $p_2^{\pi\sigma}$ are incident in $\mathrm{W}(q)$. Noting that $p_2^{\pi\sigma} = p_2^{\sigma\pi}$, we deduce that p_1 and p_2 are adjacent if and only if p_1^σ and p_2^σ are adjacent in $A(q)$. It follows that σ induces an automorphism $\overline{\sigma}$ of $A(q)$ given by $p \mapsto p^\sigma$.

Recall that the subgroup of $\mathrm{Aut}(\mathrm{W}(q))$ stabilising the set of absolute elements of π is the Suzuki group $\mathrm{Sz}(q)$. It can be easily seen that the homomorphisms from $\mathrm{Aut}(A(q))$ to $\mathrm{Sz}(q)$ and from $\mathrm{Sz}(q)$ to $\mathrm{Aut}(A(q))$ given by $\alpha \mapsto \overline{\alpha}$ and $\sigma \mapsto \overline{\sigma}$ are both injective, and hence $\mathrm{Aut}(A(q)) \cong \mathrm{Sz}(q)$.

The same argument can be used to show that $\mathrm{Aut}(I(q)) \cong \mathrm{Ree}(q)$. (Two polarities of $\mathrm{H}(q)$ with the same absolute elements coincide by [46, Corollary 7.7.3], and the group of automorphism of $\mathrm{H}(q)$ stabilising the set of absolute elements of ρ is the Ree group $\mathrm{Ree}(q)$.) □

Note that despite not being vertex-transitive, polarity graphs $B(q)$, $A(q)$ and $I(q)$ have a relatively high level of symmetry in the sense of having a fairly large automorphism group compared to their order. For example, the automorphism group of a polarity graph $A(q)$ has order $q^2(q^2+1)(q-1)$, which is almost q^2 times larger than the order q^3+q^2+q+1 of $A(q)$. In contrast, many examples of vertex-transitive graphs have the same order as their automorphism groups. (Note that the action of the automorphism group on the vertices of such graph must be regular, and hence all graphs with this property are Cayley.) This suggests that investigation of the polarity graphs can lead to constructions of large vertex-transitive, or even Cayley graphs of given degree and diameter.

In view of the constructions of finite generalised polygons in Subsection 2.1.4, it is not surprising that the structure of polarity graphs gets more complex with the increasing diameter. For this reason, our strategy is to thoroughly explore possible ways of modifying polarity graphs $B(q)$ to obtain large vertex-transitive graphs of given degree and diameter 2, and then try to apply carefully designed variants of the successful methods to $A(q)$ and $I(q)$.

3.2 Structure of polarity graphs $B(q)$

In this section we present a number of facts about polarity graphs that were first proved in [35]. We refer to [5] for much shorter proofs of these observations based on various known facts about projective planes.

Let $[\mathbf{a}]$ be a vertex of a polarity graph $B(q)$. Note that by definition of $B(q)$ it follows that $[\mathbf{a}]$ is adjacent to a vertex $[\mathbf{b}] \in B(q)$ if and only if $\mathbf{ab}^T = 0$ and $[\mathbf{a}] \neq [\mathbf{b}]$. By Proposition 2.1.9 (and its proof) we know that $[\mathbf{a}]$ has degree q or $q+1$ depending on whether $\mathbf{aa}^T$ is equal to zero or not. A vertex $[\mathbf{a}]$ satisfying $\mathbf{a}\mathbf{a}^T$ will be called a *quadric vertex* of $B(q)$. The number of quadric vertices of $B(q)$ is (again by Proposition 2.1.9) $q+1$, and hence the vertex set of $B(q)$ is a disjoint union of the set W of $q+1$ quadric vertices, and the set V of q^2 vertices of degree $q+1$. We also let V_1 denote the subset of V containing all vertices adjacent to a quadric vertex, and let $V_2 = V \setminus V_1$.

Proposition 3.2.1 ([5, Proposition 2.2]). *The graph $B(q)$ has the following properties:*

(a) *no two quadric vertices are adjacent;*

(b) *every pair of vertices in V (adjacent or not) is connected by a unique path of length 2, while no edge incident to a quadric vertex is contained in any triangle;*

(c) *the diameter of $B(q)$ is 2;*

(d) *if q is odd, then every vertex of V_1 is adjacent to exactly two quadric vertices, and the subgraphs of $B(q)$ induced by V_1 and V_2 are regular graphs of degree $(q-1)/2$ and $(q+1)/2$ and order $q(q+1)/2$ and $q(q-1)/2$, respectively;*

(e) *if q is even, then $|V_1| = q^2$ and V_2 is empty; moreover, the vertex $[1,1,1] \in V_1$ is adjacent to all quadric vertices (and no other vertex of $B(q)$), and every other vertex of V_1 is adjacent to exactly one quadric vertex; in particular, the subgraph of $B(q)$ induced by the set $V_1 \setminus \{[1,1,1]\}$ is a q-regular graph of order q^2-1.*

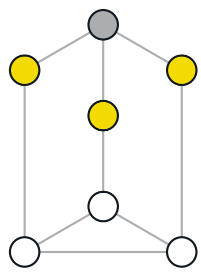
The following observation made in [17] is an easy consequence of Proposition 3.2.1.

Corollary 3.2.2. *The graph $B(q)$ has no cycle of length 4.*

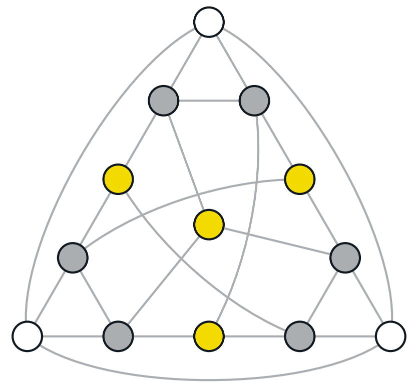
Proof. Suppose to the contrary that $B(q)$ contains a 4-cycle (u,v,w,x). Then there are two distinct uw-paths of length 2, and hence by Proposition 3.2.1(b) we find that $u \in W$ or $w \in W$. In either case, it follows by Proposition 3.2.1(a) that the neighbours v and x of u (or w) must be both contained in V. But this is impossible since v and x are connected by two distinct paths of length 2. □

The automorphism group $\mathrm{P\Gamma O}(3,q)$ of $B(q)$ obviously preserves the sets W, V_1 and V_2 for q odd, and the sets W, $\{[1,1,1]\}$ and $V_1 \setminus \{[1,1,1]\}$ for q even. In fact, it can be shown that these sets are the orbits of the vertex set of $B(q)$ under the action of the subgroup $\mathrm{PGO}(3,q)$ of $\mathrm{Aut}(B(q))$. By [24, Corollary 7.15] we know that $\mathrm{PGO}(3,q)$ has a triply transitive action on W, and the analysis of [35] shows for q odd that $\mathrm{PGO}(3,q)$ is arc-transitive on the subgraphs of $B(q)$ induced by V_1 and V_2. For q even we proved much more, extending the last observation in Section 6 of [35].

Theorem 3.2.3 ([5, Theorem 3.2]). *If q is a power of 2, then the automorphism group of the graph $B_0(q)$ induced by the set $V_1 \setminus \{[1,1,1]\}$ is isomorphic to $\mathrm{P\Gamma O}(3,q)$. Moreover, for $q \geq 4$ the smallest subgroup of $\mathrm{P\Gamma O}(3,q)$ acting transitively on the vertices of $B_0(q)$ is the group $\mathrm{PGO}(3,q)$, which also acts regularly on the arcs of $B_0(q)$. In particular, $B_0(q)$ is a vertex-transitive non-Cayley graph for $q \geq 4$.*



(a) Graph $B(2)$



(b) Graph $B(3)$

Figure 3.1: Orbits of $V(B(q))$ under the action of the automorphism group

Here we note that $\mathrm{P\Gamma O}(3,q) \cong \mathrm{P\Gamma L}(2,q)$ and $\mathrm{PGO}(3,q) \cong \mathrm{PGL}(2,q)$ for every prime power q; see [24] for example. Moreover, if q is even, then every element of $\mathrm{PGO}(3,q)$ has the form

$$\begin{pmatrix} 1+a & 1+c & 1+a+c \\ 1+b & 1+d & 1+b+d \\ 1+a+b & 1+c+d & 1+a+b+c+d \end{pmatrix} \tag{3.1}$$

with $a,b,c,d \in \mathrm{GF}(q)$ and $ad+bc=1$, and the mapping $\Phi\colon \mathrm{PGO}(3,q) \to \mathrm{PGL}(2,q)$ given by

$$\begin{pmatrix} 1+a & 1+c & 1+a+c \\ 1+b & 1+d & 1+b+d \\ 1+a+b & 1+c+d & 1+a+b+c+d \end{pmatrix} \mapsto \begin{pmatrix} c & a \\ d & b \end{pmatrix} \tag{3.2}$$

is a group isomorphism. This isomorphism will be helpful in the following section.

To conclude this section, we provide an example that illustrates some of the properties of polarity graphs $B(q)$ for both even and odd values of q.

Example 3.2.4. Figure 3.1 shows the two smallest examples of polarity graphs $B(q)$, namely $B(2)$ and $B(3)$. The three yellow vertices in Figure 3.1a correspond to quadric vertices of degree 2, the grey vertex corresponds to the vertex $[1,1,1]$ of $B(2)$, the unique vertex adjacent to all quadric vertices, and the three white vertices correspond to the vertices in $V_1 \setminus \{[1,1,1]\}$, which are adjacent to exactly one quadric vertex. By Theorem 3.1.1 we know that the automorphism group of $B(2)$ is the group $\mathrm{P\Gamma O}(3,2)$, which is isomorphic to the symmetric group $\mathrm{Sym}(3)$. The three colours of vertices in Figure 3.1a correspond to the three orbits of $V(B(2))$ under the action of $\mathrm{P\Gamma O}(3,2)$. Note that by Theorem 3.2.3 this is the only case for q even when the subgraph $B_0(q)$ of $B(q)$ induced by the set $V_1 \setminus \{[1,1,1]\}$ is Cayley. In particular, $B_0(2)$ is isomorphic to the circulant $C(\mathbb{Z}_3, \{1,2\})$.

Similar to the previous case, the yellow vertices in Figure 3.1b correspond to the quadric vertices in W of degree 3. The set V of 9 vertices of degree 4 is a disjoint union of the set V_1 containing the vertices adjacent to a quadric vertex, and the set V_2 of vertices with no neighbour in W. Vertices in sets V_1 and V_2 correspond to the grey and white vertices in Figure 3.1b.

The automorphism group of $B(3)$ is (again by Theorem 3.1.1) the group $\mathrm{P\Gamma O}(3,3)$, which is isomorphic to the symmetric group $\mathrm{Sym}(4)$. Again the three colours of vertices in Figure 3.1b correspond to the three orbits of $V(B(3))$ under the action of $\mathrm{P\Gamma O}(3,3)$. Note that the subgraph induced by the grey vertices is disconnected. With some work, it can be shown that this is the only case for q odd when at least one of the subgraphs of $B(q)$ induced by V_1 and V_2 is disconnected. In contrast, by Proposition 3.2.1(a) we know that for every prime power q the subgraph of $B(q)$ induced by W is totally disconnected.

3.3 Vertex-transitive graphs from polarity graphs $B(q)$?

Polarity graphs $B(q)$ can be easily extended to $(q+1)$-regular graphs by adding a perfect matching between the vertices in W for q odd, and by adding an extra vertex adjacent to all quadric vertices for q even. Even though these graphs are regular, they are not vertex-transitive. In the case when q is even, this follows by the fact that the vertex $[1,1,1]$ and the new vertex are the only two vertices with the same neighbourhood. In the case when q is odd, this is an easy consequence of Proposition 3.2.1; see [42]. In [5] we extended the latter observation by showing the following:

Theorem 3.3.1 ([5, Theorem 4.1])**.** *For any odd prime power $q \geq 37$ there is no vertex-transitive graph of degree $q+3$ which contains the polarity graph $B(q)$ as a spanning subgraph.*

In Theorem 3.3.1 and the discussion above, we addressed the question if we can add "a few" edges to $B(q)$ to obtain a vertex-transitive graph. On the other hand, some interesting vertex-transitive graphs can be obtained as induced subgraphs of polarity graphs $B(q)$. For example, by Proposition 3.2.1 and Theorem 3.2.3 we know that for every even prime power q the subgraph $B_0(q)$ of $B(q)$ induced by the set $V_1 \setminus \{[1,1,1]\}$ is a vertex-transitive non-Cayley graph of degree q and order q^2-1. For q odd, the subgraphs $B_1(q)$ and $B_2(q)$ induced by V_1 and V_2 are vertex transitive graphs of degree $(q-1)/2$ and $(q+1)/2$ and order $q(q+1)/2$ and $q(q-1)/2$, respectively.

In contrast with Theorem 3.2.3, if q is even, then there exists a subgroup of $\mathrm{PGO}(3,q)$ that is regular on the set $V^* = V_1 \setminus \{[t,t,1], t \in \mathrm{GF}(q)\}$. Namely, if H is the subgroup of $\mathrm{PGO}(3,q)$ formed by the matrices of the form (3.1) such that $a+b+c+d=0$, then it can be easily verified that $|H| = q(q-1)$, and that H is regular on the V^*. In particular, the subgraph $B^*(q)$ of $B(q)$ induced by the set V^* is a Cayley graph for H. The following theorem shows that, quite surprisingly, the Cayley graphs $C(G,C)$ from Construction 2.2.7 are isomorphic to the graphs $B^*(q)$.

Theorem 3.3.2 ([5, Theorem 4.3])**.** *If q is a power of 2, then $C(G,C)$ is isomorphic to $B^*(q)$.*

4 Cayley graphs of diameter 2 and 3

In this chapter we show that the Moore bound can be asymptotically approached for diameters 2 and 3 using Cayley graphs obtained by extending regular orbits of suitable subgroups of $\mathrm{Aut}(B(q))$ and $\mathrm{Aut}(A(q))$. We also show that both constructions are based on a more general unifying principle.

4.1 Cayley graphs from $B(q)$

Recall that for q even we define $B^*(q)$ as the subgraph of the polarity graph $B(q)$ induced by the set $V_1 \setminus \{[t, t, 1], t \in \mathrm{GF}(q)\}$. The main theorem of this section is the following:

Theorem 4.1.1 ([5, Theorem 4.2]). *For every even prime q there exists a Cayley graph of diameter 2 and degree $q + O(\sqrt{q})$ as $q \to \infty$, with $B^*(q)$ as a spanning subgraph.*

Proof. Let $F = \mathrm{GF}(q)$, let G be the one-dimensional affine group over F represented as the semidirect product $G = F^+ \rtimes F^*$ with the multiplication given by $(a, b)(c, d) = (a + bc, bd)$, and let C be the subset of G consisting of all elements of the form (c, c^2), where c is a non-zero element of F. Then there exist sets A and B (described in Construction 2.2.7) such that $C(G, A \cup B \cup C)$ is a Cayley graph of diameter 2, and $|A \cup B| = O(\sqrt{q})$ as $q \to \infty$.

Next, let H be the subgroup of $\mathrm{PGO}(3, q)$ formed by the matrices of the form (3.1) with $a + b + c + d = 0$, let X be a generating set for H such that $C(H, X)$ is isomorphic to $B^*(q)$, and let φ be an isomorphism from H to G which maps X to C. (An example of such isomorphism is given in Theorem 3.3.2.) Then $B^*(q)$ is a spanning subgraph of the Cayley graph $C(H, \varphi^{-1}(A) \cup \varphi^{-1}(B) \cup X)$ of degree $q + O(\sqrt{q})$ and diameter 2. □

We note that the original proof of Theorem 4.1.1 presented in [5] is different from the one presented here, and it uses various properties of polarity graphs $B(q)$ for q even.

4.2 Cayley graphs from $A(q)$

In [4] we showed that a variant of the method used in the original proof of Theorem 4.1.1 can be applied to polarity graphs $A(q)$, giving us the following:

Theorem 4.2.1 ([4, Theorem 5.1]). *For every positive integer e and $q = 2^{2e+1}$ there exists a Cayley graph of diameter 3 and degree $q + O(\sqrt{q})$ as $q \to \infty$.*

The following fact is an immediate consequence of Theorem 4.2.1.

Corollary 4.2.2. *The Moore bound can be asymptotically approached for diameter 3 by Cayley graphs.*

4.3 General principle

In [3] we showed that the methods we described in Sections 4.1 and 4.2 are based on a certain underpinning principle which we reproduce here.

Theorem 4.3.1 ([3, Theorem 2.1]). *Let Γ be a graph of maximum degree d and diameter $k \geq 2$ such that:*

(1) *there is a subgroup G of $\mathrm{Aut}(\Gamma)$ regular on one of its orbits $O \subset V$;*

(2) *every vertex $v \in V \setminus O$ adjacent to a vertex in O has a vertex stabiliser in G that acts regularly on $N(v) \cap O$; and*

(3) *there exists a $\delta > 0$ with the property that for any pair of vertices $u, v \in O$ there is a shortest uv-path P in Γ such that every vertex P has at least $d - \delta$ neighbours in O.*

Then, letting $\gamma = \gamma(\delta, k) = \delta + \delta(\delta - 1) + \delta(\delta - 1)^{k-2}$, there exists a Cayley graph for G of degree at most $d - \delta + \gamma(5\sqrt{d} + 2)$, and diameter at most k.

Theorem 4.3.1 can be applied to reprove Theorems 4.1.1 and 4.2.1; see [3, Section 3]. In both cases the proofs are considerably shorter than the original ones [5, 4], but (due to a larger generality of Theorem 4.3.1) the bounds on generating sets of the corresponding Cayley graphs are slightly worse in the $O(\sqrt{q})$ term. Nevertheless, the resulting Cayley graphs still approach the Moore bound for diameters 2 and 3.

5 Generalisations and future work

In this chapter we explain why does the approach that works for construction of larger Cayley graphs of diameter 2 and 3 does not carry over to an analogous construction of Cayley graphs of diameter 5 from split Cayley hexagons. We also discuss possible future directions of research, and we formulate some open problems.

5.1 Polarity graphs $I(q)$

In the previous chapter we saw that polarity graphs $B(q)$ and $A(q)$ can be used to construct two families of Cayley graphs that asymptotically approach the Moore bound for diameters 2 and 3. In this section we look at polarity graphs $I(q)$, the remaining of the three classes of polarity graphs introduced in Subsection 2.1.5.

5.1.1 Cayley graphs of diameter 5

Let $q = 3^{2e+1}$ for some non-negative integer e, and recall that the polarity graph $I(q)$ has maximum degree $q+1$, diameter 5, and order $q^5+q^4+q^3+q^2+q+1$, and that $\mathrm{Aut}(I(q)) \cong \mathrm{Ree}(q)$. It is tempting to consider the same approach as in [5] and [4] for constructing an infinite sequence of Cayley graphs of degree $q + o(q)$, diameter 5, and order $q^5 - o(q^5)$ as $q \to \infty$. The first (and most important) step of this approach is to find a subgroup G of $\mathrm{Aut}(I(q))$ regular on one of its orbits of size $q^5 - o(q^5)$. Unfortunately, by the classification of maximal subgroups of Ree groups (see [27, 26]), we know that $\mathrm{Ree}(q)$ has no subgroup of order $O(q^5)$ for $q \to \infty$, and thus the approach that works for diameters 2 and 3 does not carry over to an analogous construction for diameter 5.

5.1.2 Cayley graphs of diameter 4

Unlike the case of $\mathrm{Aut}(I(q))$, the automorphism groups of $B(q)$ (for $q = 2^e$) and $A(q)$ contain a subgroup of order $q^k - o(q^k)$, where k is the diameter of the corresponding polarity graph. In the case of polarity graphs $B(q)$, this subgroup is isomorphic to the one-dimensional affine group over $F = \mathrm{GF}(q)$ represented as the semidirect product $G = F^+ \rtimes F^*$ with the multiplication given by $(a, b)(c, d) = (a + bc, bd)$. In the case of polarity graphs $A(q)$, this subgroup is formed by 4×4 matrices $M(r; a, b)$ defined in [4]. Both subgroups, which we denote by $G_2(q)$ and $G_3(q)$, can be defined as a subgroup of the automorphism group of the corresponding polarity graph stablising a vertex of degree q.

Since both $G_2(q)$ and $G_3(q)$ are underlying groups of Cayley graphs that asymptotically approach the Moore bound for diameters 2 and 3, it is natural to look at an analogously defined subgroup of $\mathrm{Ree}(q)$. Namely, we are interested in the group $G_4(q)$ isomorphic to a subgroup of $\mathrm{Aut}(I(q))$ stabilising a vertex of degree q. (Again, all such subgroups are conjugate to each other, and hence isomorphic.) Equivalently, $G_4(q)$ is isomorphic to a stabiliser in $\mathrm{Ree}(q)$ of an absolute point of the polarity ρ (defined in Subsection 2.1.4). It turns out that $G_4(q)$ has order $q^3(q-1)$, the elements of $G_4(q)$ can be represented by quadruples $(r; a, b, c)$ with $r \in F^*$ and $a, b, c \in F$, and the multiplication is given by $(r; a, b, c)(s; x, y, z) = (rs; as + x, ax^\theta s + bs^{\theta+1} + y, ays + cs^{\theta+2} + z - ax^{\theta+1}s - bxs^{\theta+1})$, where $q = 3^{2e+1}$ and θ is the Tits isomorphism of $\mathrm{GF}(q)$. For further details, including the action of the elements of $G_4(q)$ on the points and lines of the split Cayley hexagon $\mathrm{H}(q)$, we refer the reader to [46, Item 7.7.7].

The underlying reason why it is possible to construct Cayley graphs of degree $q + o(q)$ and diameter 2 or 3 for $G_2(q)$ and $G_3(q)$ is the following property of these groups.

Proposition 5.1.1. *For each $k \in \{2, 3\}$ and suitable q there exists a generating set S of order $q - 1$ for $G_k(q)$ such that S is closed under taking inverses, $1 \notin S$, and all but $o(q^k)$ elements of $G_k(q)$ can be written as a product of k elements of S for $q \to \infty$.*

Given the obvious connection between the groups $G_2(q)$, $G_3(q)$ and $G_4(q)$, it is natural to ask whether the analogue of Proposition 5.1.1 also holds for the latter:

Question 5.1.2. Does there exists a generating set S of order $q - 1$ for $G_4(q)$ such that S is closed under taking inverses, $1 \notin S$, and all but $o(q^4)$ elements of $G_4(q)$ can be written as a product of 4 elements of S as $q \to \infty$?

If one can find such set S, it could potentially lead to construction of Cayley graphs that asymptotically approach the Moore bound for diameter 4. Note that this would also solve Delorme's problem for diameter 4; as we mentioned in Subsection 2.2.3, the best available lower bound on Delorme's limit superior for general graphs of diameter 4 is $1/4$. Also note that it would be equally interesting (from an asymptotic point of view) to find a set S with given properties of size $q + o(q)$, but in view of Proposition 5.1.1 we believe that the most promising strategy is to restrict ourselves to generating sets of size $q - 1$.

A slightly tedious but straightforward method to find a suitable generating set S for the group $G_k(q)$ is to find an induced subgraph (of the corresponding polarity graph) isomorphic to a Cayley graph $C(G, X)$, and take $S = X$. Unfortunately, this can be done only if the corresponding polarity graph contains an induced Cayley subgraph of degree $q - 1$ and order $|G|$, which is impossible (even if we drop the assumption that the subgraph is induced and Cayley) in the case when $G = G_4(q)$:

Lemma 5.1.3. *For any sufficiently large q the polarity graph $I(q)$ contains no subgraph of degree $q - 1$ and order $q^3(q - 1)$.*

Note that Proposition 5.1.1 for $k = 2$ can be also proved by showing that the following system of equations (which appeared in Construction 2.2.7) has a solution $(x, y) \in F^* \times F^*$ for all but $o(q^2)$ elements $(r, s) \in G_2(q)$:

$$r = x + x^2 y,$$
$$s = x^2 y^2.$$

Similarly, it can be shown that for all but $o(q^3)$ elements of $G_3(q)$, represented by matrices $M(x;y,z)$, there exist elements $r,s,t \in F^*$ such that $M(x;y,z) = M(r;a(r),1)M(s;a(s),1)$ $M(t;a(t),1)$ or, equivalently, such that

$$\begin{aligned} x &= rst, \\ y &= a(r)st + a(s)t + a(t), \\ z &= (a(r)st + a(s)t)(a(s)t + a(t))^{\omega} + 1, \end{aligned}$$

which proves Proposition 5.1.1 for $k = 3$. This proof is unsurprisingly much more difficult than in the case when $k = 2$; see [2] for details.

The previous paragraph provides another perspective on how to approach Question 5.1.2. Namely, we can choose some generating set S for $G_4(q)$, and try to show that all but $o(q^4)$ elements of $G_4(q)$ can be written as a product of 4 elements of S. This is equivalent to showing that a certain system of four equations has a solution in all but $o(q^4)$ cases. Using inspiration from the case of $G_3(q)$, it is possible to find a few candidates for S. (By a candidate we mean a generating set with $q-1$ elements which is closed under taking inverses and does not contain the identity element.) So far, however, we do not know any reliable method to determine for which elements of $G_4(q)$ the corresponding system of equations has a solution.

5.1.3 General graphs of diameter 4 or 5

All known families of graphs that asymptotically approach the Moore bound for some diameter are either polarity graphs $B(q)$, $A(q)$ and $I(q)$, or their modifications. Hence an obvious next step is to investigate discrete structures similar to these families of graphs. Since we already know that the Moore bound can be asymptotically approached for diameters 2 and 3 by general, vertex-transitive, and even Cayley graphs, we are interested only in structures similar to polarity graphs $I(q)$. One possible approach is to look at the structures related to the automorphism group of $I(q)$, which is isomorphic to the Ree group $\mathrm{Ree}(q)$. Two such structures are Ree geometries [23] and Ree unitals [29], latter of which we describe below.

Let $G = \mathrm{Ree}(q)$, let P be the set of all q^3+1 Sylow 3-subgroups of G, and consider the conjugation action of G on P. It can be shown that each involution of G has $q+1$ fixed points, and that for any two distinct points there is a unique involution of G fixing both of them. Hence we can construct a block design on the points of P whose blocks are the sets of fixed points for each involution of G. The resulting 2-$(q^3+1, q+1, 1)$ block design is called a Ree unital. We hope that a thorough investigation of Ree unitals will help us to find a new family of (possibly Cayley) graphs that asymptotically approach the Moore bound for diameter 5, or maybe even 4. For further information on Ree unitals we refer the reader to [1].

5.2 Degree-diameter problem and generalised octagons

Recall that finite generalised n-gons exist only for $n = 3$, 4, 6 or 8. As we have seen in the previous chapters, generalised triangles, quadrangles and hexagons play a prominent role in the degree-diameter problem. In contrast, there is not a single known construction (relevant to this problem) based on generalised octagons. In this section we briefly explain a reason behind this.

The only known finite generalised octagons are the Ree-Tits octagons $\mathrm{O}(q)$ of order (q, q^2), where q is an odd power of two, and their duals. (The dual of a geometry is a geometry obtained by interchanging the roles of points and lines.) The Ree-Tits octagon $\mathrm{O}(q)$ has $q^{10}+q^9+\cdots+1$ points and $q^{11}+q^{10}+\cdots+1$ lines, and so the incidence graph $L(q)$ of $\mathrm{O}(q)$ is a bipartite graph of maximum degree q^2+1 on $q^{11}+2q^{10}+2q^9+\cdots+2$ vertices. Moreover, by Lemma 2.1.4 we know that $L(q)$ has diameter 8. There is no obvious way to turn this graph into a graph of diameter 7 and order $q^7+o(q^7)$. Even if we succeeded, it seems to be almost certain that the maximum degree in the resulting graph would be at least q^2+1. (In which case, this graph is not interesting for the degree-diameter problem.) Nevertheless, we believe that Ree-Tits octagons are worth of investigation in the context of the degree-diameter problem. One potential approach is to examine some other family of finite generalised polygons of order (q, q^2) with simpler structure, and see if we can use it to construct large graphs of given maximum degree and diameter. A good candidate for this is the family of orthogonal quadrangles $\mathrm{Q}(5, q)$; see [46, Section 2.3].

5.3 Vertex-transitive closure of a graph

In Section 3.3 we saw that for q odd the polarity graph $B(q)$ cannot be extended (by adding extra edges) to a vertex-transitive graph of degree $q+1$ or $q+3$. An interesting question is what is the smallest integer d for which there exists a d-regular vertex-transitive graph which contains $B(q)$ as a spanning subgraph. This leads to the following more general definition.

Definition 5.3.1. A *vertex-transitive closure* of a graph Γ is a vertex-transitive supergraph of Γ on the same vertex set. A *vertex-transitive number* of a graph Γ, denoted by $d_{vt}(\Gamma)$, is the smallest integer for which there exists a $d_{vt}(G)$-regular vertex-transitive closure of Γ.

We have the following for polarity graphs $B(q)$ with q odd.

Theorem 5.3.2. *For any odd prime power $q \geq 37$ the vertex-transitive number of $B(q)$ is at least $q+5$.*

The problem of determining (or at least estimating) the vertex-transitive number of $B(q)$ appears to be quite interesting in the context of the degree-diameter problem. In particular, if one can show that $d_{vt}(B(q)) = q + o(q)$, then this gives a new family of vertex-transitive graphs that asymptotically approach the Moore bound for diameter 2. There are many other interesting questions related to vertex-transitive closures of graphs, and even though they are mostly not related to the degree-diameter problem, we include a few of them here.

Question 5.3.3. Is there a constant m such that every planar graph of maximum degree d has a vertex-transitive closure of degree at most md?

Question 5.3.4. What is the smallest positive integer d_n for which there exists a graph Γ of order n such that the maximum degree in Γ is d_n, and the only vertex transitive closure of Γ is the complete graph K_n?

Bibliography

[1] E.F. Assmus, Jr. and J.D. Key, Arcs and Ovals in the Hermitian and Ree Unitals, *European J. Combin.* 10 (1989), 297–308.

[2] M. Bachratý, Cayley graphs and degree-diameter problem for diameter 3, in: *Advances in Architectural, Civil and Environmental Engineering*, Slovak University of Technology in Bratislava, 2016, 20–24.

[3] M. Bachratý, J. Šiagiová and J. Širáň, Revisiting Constructions of Large Cayley Graphs of Given Degree and Diameter from Regular Orbits, *J. Interconnect. Netw.* 17 (2017), 1–8.

[4] M. Bachratý, J. Šiagiová and J. Širáň, Asymptotically approaching the Moore bound for diameter three by Cayley graphs, *J. Combin. Theory Ser. B* 134 (2019), 203–217.

[5] M. Bachratý and J. Širáň, Polarity graphs revisited, *Ars Math. Contemp.* 8 (2015), 55–67.

[6] R. Baer, Polarities in finite projective planes, *Bull. Amer. Math. Soc.* 52 (1946), 77–93.

[7] E. Bannai and T. Ito, On finite Moore graphs, *J. Fac. Sci. Tokyo Univ.* 20 (1973), 191–208.

[8] J.-C. Bermond and B. Bollobás, The diameter of graphs: a survey, *Congressus Numerantium* 32 (1981), 3–27.

[9] D. Bevan, G. Erskine and R. Lewis, Large circulant graphs of fixed diameter and arbitrary degree, *Ars Math. Contemp.* 13 (2017), 275–291.

[10] E. Canale and J. Gómez, Superfluous edges and exponential expansions of de Bruijn and Kautz digraphs, *Discrete Appl. Math.* 138 (2004), 303–331.

[11] Combinatorics Wiki: The degree diameter problem for general graphs, accessed March 2021.

[12] C. Dalfó, A survey on the missing Moore graph, *Linear Algebra Appl.* 569 (2019), 1–14.

[13] R.M. Damerell, On Moore graphs, *Math. Proc. Camb. Philos. Soc.* 74 (1973), 227–236.

[14] V. de Smet and H. van Maldeghem, The finite Moufang hexagons coordinatized, *Beitr. Algebra Geom.* 34 (1993), 217–232.

[15] C. Delorme, Large bipartite graphs with given degree and diameter, *J. Group Theory* 8 (1985), 325–334.

[16] C. Delorme, Examples of products giving large graphs with given degree and diameter, *Discrete Appl. Math.* 37/38 (1992), 157–167.

[17] P. Erdős and A. Rényi, On a problem in the theory of graphs, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 7 (1962), 623–641.

[18] G. Erskine, Diameter 2 Cayley graphs of dihedral groups, *Discrete Math.* 338 (2015), 1022–1024.

[19] G. Exoo, R. Jajcay, M. Mačaj and J. Širáň, On the defect of vertex-transitive graphs of given degree and diameter, *J. Combin. Theory Ser. B* 134 (2019), 322–340.

[20] V. Faber, J.W. Moore and W.Y.C. Chen, Cycle prefix digraphs for symmetric interconnection networks, *Networks* 23 (1993), 641–649.

[21] W. Feit and G. Higman, The nonexistence of certain generalized polygons, *J. Algebra* 1 (1964), 114–131.

[22] M.J. Ganley, Regular Polarities of Finite Projective Planes, *Bull. London Math. Soc.* 3 (1971), 209–214.

[23] F. Haot, K. Struyve and H. van Maldeghem, Ree geometries, *Forum Math.* 23 (2011), 75–98.

[24] J. Hirschfeld, *Projective geometries over finite fields*, Oxford University Press, 1998.

[25] A.J. Hoffman and R.R. Singleton, On Moore graphs with diameter 2 and 3, *IBM J. Res. Develop.* 4 (1960), 497–504.

[26] P.B. Kleidman, The maximal subgroups of the chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups, *J. Algebra* 117 (1988), 30–71.

[27] V.M. Levchuk and Y.N. Nuzhin, Structure of Ree groups, *Algebra Log.* 24 (1985), 16–26.

[28] E. Loz and J. Širáň, New record graphs in the degree-diameter problem, *Australas. J. Combin.* 41 (2008), 63–80.

[29] H. Lüneburg, Some remarks concerning the Ree groups of type (G_2), *J. Algebra* 3 (1966), 256–259.

[30] H. Macbeth, J. Šiagiová and J. Širáň, Cayley graphs of given degree and diameter for cyclic, Abelian, and metacyclic groups, *Discrete Math.* 312 (2012), 94–99.

[31] H. Macbeth, J. Šiagiová, J. Širáň and T. Vetrík Large Cayley graphs and vertex-transitive non-cayley graphs of given degree and diameter, *J. Graph Theory* 64 (2010), 87–98.

[32] M. Mačaj and J. Širáň, Search for properties of the missing Moore graph, *Linear Algebra Appl.* 432 (2010), 2381–2398.

[33] M. Miller and J. Širáň, Moore graphs and beyond: a survey of the degree/diameter problem, *Electron. J. Combin.* 20 (2013), Dynamic Survey DS14v2, 92 pp.

[34] U. Ott, Eine Bemerkung über Polaritäten eines verallgemeinerten Hexagons, *Geom. Dedicata* 11 (1981), 341–345.

[35] T.D. Parsons, Graphs from projective planes, *Aequat. Math.* 14 (1976), 167–189.

[36] S.E. Payne, Symmetric representations of nondegenerate generalized n-gons, *Proc. Amer. Math. Soc.* 19 (1968), 1321–1326.

[37] S.E. Payne, *Topics in Finite Geometry: Ovals, Ovoids, and Generalized Quadrangles*, University of Colorado, Denver, 2007.

[38] G. Sabidussi, On a Class of Fixed-Point-Free Graphs *Proc. Am. Math. Soc.* 9 (1958), 800–804.

[39] G.L. Schellekens, On a hexagonic structure, I, *Indag. Math.* 24 (1962), 201–217.

[40] G.L. Schellekens, On a hexagonic structure, II, *Indag. Math.* 24 (1962), 218–234.

[41] J. Šiagiová and J. Širáň, Approaching the Moore bound for diameter two by Cayley graphs, *J. Combin. Theory Ser. B* 102 (2012), 470–473.

[42] J. Širáň, J. Šiagiová and M. Ždímalová, Large graphs of diameter two and given degree, in: *Proc. IWONT 2010*, Univ. Politecnica de Catalunya, 2011, 347–359.

[43] J. Tits, Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* 2 (1959), 13–60.

[44] J. Tits, Ovoïdes et Groupes de Suzuki, *Arch. Math.* 13 (1962), 187–198.

[45] J. Tits, *Buildings of Spherical Type and Finite BN-Pairs*, Lecture Notes in Mathematics 386, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[46] H. van Maldeghem, *Generalized Polygons*, Monographs in Mathematics 93, Birkhäuser Basel, Basel, 1998.

[47] T. Vetrík, Cayley graphs of given degree and diameters 3, 4 and 5, *Discrete Math.* 313 (2013), 213–216.

[48] T. Zhang and G. Ge, Improved lower bounds on the degree–diameter problem, *J. Algebr. Combin.* 49 (2019), 135–146.

List of publications

M. Bachratý and J. Širáň, Polarity graphs revisited, *Ars Math. Contemp.* 8 (2015), 55–67.

M. Bachratý, Asymptotically approaching the Moore bound by Cayley graphs, in: *Advances in Architectural, Civil and Environmental Engineering*, Slovak University of Technology in Bratislava, 2015.

M. Bachratý, Asymptotically approaching the Moore bound for diameters two and three, in: *Mathematics in Science and Technologies 2016: proceedings of the MIST conference*, 2016.

M. Bachratý, Cayley graphs and degree-diameter problem for diameter 3, in: *Advances in Architectural, Civil and Environmental Engineering*, Slovak University of Technology in Bratislava, 2016.

M. Bachratý and R. Jajcay, Classification of coset-preserving skew-morphisms of finite cyclic groups, *Australas. J. Combin.* 67 (2017), 259–280.

M. Bachratý and R. Jajcay, Powers of skew-morphisms, in: *Symmetries in Graphs, Maps, and Polytopes*, Springer (2016), 1–25.

M. Bachratý, J. Šiagiová and J. Širáň, Revisiting Constructions of Large Cayley Graphs of Given Degree and Diameter from Regular Orbits, *J. Interconnect. Netw.* 17 (2017), 1–8.

M. Bachratý, J. Šiagiová and J. Širáň, Asymptotically approaching the Moore bound for diameter three by Cayley graphs, *J. Combin. Theory Ser. B* 134 (2019), 203–217.

M. Bachratý, M. Conder and G. Verret, Skew product groups for monolithic groups, submitted for publication.

V. Bachratá and **M. Bachratý**, Orientably-regular maps of given hyperbolic type with no non-trivial exponents, submitted for publication.