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Summary of PhD dissertation

Graph coverings in the degree-diameter problem and in the degree-girth problem

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Abstrakt

V teórii grafov medzi rozsiahlo študované problémy patrí problém stupňa a priemeru, ako aj problém stupňa a obvodu. V prvom prípade je úlohou nájsť grafy maximálneho rádu $n_{d,k}$, ktoré majú daný maximálny stupeň vrcholu d a daný priemer k. V druhom prípade ide o nájdenie grafov minimálneho rádu $n_{[d,g]}$, pričom minimálny stupeň vrchola je d a obvod grafu je g. V obidvoch úlohách sú známe teoretické hranice, ktoré nie je možné prekročiť, tzv. moorovské hranice. Bolo dokázané, že tieto hranice sa nadobúdajú iba pre zriedkavé hodnoty dvojíc (d, k) a $[d, g]$. Vzhľadom na tento skutkový stav sa rozsiahlo študujú rôzne konštrukcie 'veľkých' grafov daného stupňa a priemeru na jednej strane a 'malých' grafov daného stupňa a obvodu na strane druhej.

V našej práci sme sa sústredili na špeciálny typ takých konštrukcií, a to pomocou regulárnych nakrytí grafov. Práca obsahuje tri typy výsledkov. Po prehľade aktuálnych poznatkov uvádzame nové horné odhady pre rády grafov daného stupňa a priemeru dva, skonštruovaných pomocou zdvihov istých špeciálnych typov grafov s napäťovými priradeniami v abelovských grupách. Neskôr sa v práci venujeme klietkam obvodu 6 a im príbuzným grafom a ukazujeme, že tieto grafy sú regulárnymi zdvihmi dipólov. Podobným spôsobom tiež študujeme grafy príbuzné klietkam obvodu 8 spolu s relevantnými maticovými grupami a súvislosťami s tzv. G-grafmi. Napokon pomocou analýzy grúp automorfizmov ukazujeme, že istá ďalšia trieda grafov odvodených od klietok obvodu 8 nepripúšťa konštrukciu pomocou zdvihov dipólov.

Obsah

1 Introduction

Interest in finding and construction of graphs with given properties dates to 1950's, where communication networks and data organization or the flow of computation were modelled by various types of graphs. The limited hardware and resources in building such a network helped to form two of the well known problems in extremal graph theory, namely the degree-diameter problem and the degree-girth problem.

The aim of the degree-diameter problem is to find graphs of largest order $n_{d,k}$ subject to maximum degree d and diameter k. The degree-girth problem deals with finding the smallest order $n_{[d,g]}$ of a graph of minimum degree d and girth g. In both problems there are known theoretical bounds that can not be exceeded, called the Moore bounds. It is also known that these bounds can be achieved only for rare pairs of (d, k) and $[d, g]$. Due to these facts there emerged a stream of research into diverse construction methods to produce 'large' graphs of given degree and diameter on the one hand, and 'small' graphs of given degree and girth on the other hand.

In our dissertation we focused on special types of such constructions, namely, on regular coverings of graphs, presenting three main types of results. We begin with an overview of current results in Chapter 2. In Chapter 3 we provide new upper bounds on orders of graphs in the degree-diameter problem for diameter two, constructed by lifting special types of graphs by means of voltage assignments in Abelian groups. Chapter 4 is devoted to cages and near-cages of girth 6 and we show there that these graphs arise as regular lifts of dipoles. In Chapter 5 we similarly study near-cages of girth 8 together with relevant matrix groups and the correspondence with the so-called G-graphs. Finally, in Chapter 6 with the help of an analysis of automorphism groups we show that a known class of graphs derived from cages of girth 8 are not isomorphic to any regular lifts of dipoles.

2 Background

2.1 The degree-diameter problem

For given integers d and k, the *degree-diameter problem* is to determine the largest number of vertices $n_{d,k}$ in a graph (called the order of a graph) of maximum degree d and given diameter k, and such a graph will be called a (d, k)-graph. This problem was introduced in 1960 by A. J. Hoffman and R. R. Singleton [21] and since then, research in this area has taken two main approaches:

- Proofs of non-existence of graphs of order close to the Moore bound - leading to upper bounds on the $n_{d,k}$.
- Constructions of large graphs for given d and k - furnishing better lower bounds on $n_{d,k}$.

The Moore bound

There is a straightforward upper bound on the largest possible order of a (d, k)-graph, called the *Moore bound* and denoted by $M(d, k)$, which arises from spanning tree structure. If $d = 1$ then $k = 1$ and $M(1, 1) = 2$; the order of a star, with central vertex of degree d, is $M(d, 1) = d+1$ for all $d \geq 1$; and the order of a path of length $2k$ is $M(2, k) = 2k + 1$ for all $k \geq 1$. Then for any $d \geq 3$ and $k \geq 2$ there is the following formula

$$M(d, k) = 1 + d + d(d-1) + \ldots + d(d-1)^{k-1} = 1 + d\frac{(d-1)^k - 1}{d-2} \ . \tag{2.1}$$

For the largest order $n_{d,k}$ of a (d, k)-graph it holds that $n_{d,k} \leq M(d, k)$ for all $d, k \geq 1$.

Graphs for which $n_{d,k} = M(d, k)$ were called in [21] the *Moore graphs*, and it is known that they are necessarily regular of degree d. The authors in [21] also proved that for diameter $k = 2$ Moore graphs exist only for $d \in \{2, 3, 7\}$, excluding all other values of $d \geq 2$ except $d = 57$. For diameter 3 they showed that the unique Moore graph is the 7-cycle.

The absence of Moore (d, k)-graphs for all but a few values of d, k was proved by Damerell [12] and Bannai with Ito [5], which independently and using different approaches demonstrated that no Moore (d, k)-graphs exist for $d \geq 3$ and $k \geq 3$.

Bounds in the degree-diameter problem

The well known *de Bruijn* digraphs give the lower bound $n_{d,k} \geq (d/2)^k$. The *Kautz graphs* [22] imply for $k \geq 3$ and even $d \geq 4$ a better lower bound

$$n_{d,k} \geq \left(\frac{d}{2}\right)^k + \left(\frac{d}{2}\right)^{k-1} \ .$$

Further improved comes from Canale and Gómez [11] to

$$n_{d,k} \geq \left(\frac{d}{1,57}\right)^k \ ,$$

for k congruent to $-1, 0$, or 1 (mod 6) and an infinite set of degrees for each such k.

For diameter 2 remarkable lower bound follows from Brown graphs [10], which have maximum degree $q+1$, q a prime power

$$n_{d,2} \geq d^2 - d + 1$$

for each d such that $d-1$ is a prime power. By [14, 13] it is known that this can be improved by adding 1 to the right-hand side in the case when q is a power of 2. The bound can be extending by steadily improving number-theoretic results about distributions of primes to $n_{d,2} \geq d^2 - o(d^2)$ with a smaller and smaller $o(d^2)$ term.

Vertex-transitive and Cayley (d,k)-graphs

For $d \geq 3$ and $k \geq 2$ let $vt_{d,k}$ and $Cay_{d,k}$ be the largest order of a vertex-transitive and a Cayley graph, respectively, of degree d and diameter k.

The previous facts imply that in this range of d and k the only Moore graphs that are vertex-transitive are the Petersen and the Hoffman-Singleton graph, which implies that $vt_{3,2} = 10 = M(3,2)$ and $vt_{7,2} = 50 = M(7,2)$, while $vt_{d,k} < M(d,k)$ for all the remaining pairs (d,k) in our range. It is also known by G. Exoo, R. Jajcay, M. Mačaj and J. Širáň [17] that for any fixed $d \geq 3$ and an arbitrarily large positive integer c it holds that $vt_{d,k} \leq M(d,k) - c$ for almost all diameters k.

For Cayley graphs the current asymptotically best results are due to J. Šiagiová and J. Širáň [31] for diameter 2 by proving that $Cay_{d,2} \geq d^2 - o(d^2)$ for an infinite increasing sequence of degrees d, and to M. Bachratý, J. Šiagiová and J. Širáň [3] by showing that $Cay_{d,3} \geq d^3 - o(d^3)$ for another sequence of degrees.

Considering vertex-transitive but non-Cayley graphs the current records were set by McKay, Miller and Širáň [25] by showing that for all degrees d of the form $(3q-1)/2$, where q is a prime power congruent to 1 (mod 4), there are vertex-transitive non-Cayley graphs of diameter 2, degree d and order

$$\frac{8}{9}\left(d + \frac{1}{2}\right)^2 . \tag{2.2}$$

2.2 The degree-girth problem

For given integers d and g, the *degree-girth problem* is to find the smallest number of vertices $n_{[d,g]}$ in a graph of minimum degree d and given girth g, and such a graph will be called a $[d,g]$-graph. Then a $[d,g]$-graph of the smallest possible order is a $[d,g]$-*cage*. Introduction of this problem is dated to 1947 by W. T. Tutte [32], and a summary of the development can be found in the dynamic survey [15] by G. Exoo and R. Jajcay.

Bounds in the degree-girth problem

Lower bounds on $n_{[d,g]}$ are obtained in a way similar to the upper bound in the degree-diameter problem, but one needs to consider parity of the girth. If g is odd, the lower bound on the order of $[d,g]$-graph is:

$$n_{[d,g]} \geq 1 + d\frac{(d-1)^{(g-1)/2} - 1}{d-2} \quad \text{for } d \geq 3 \text{ and odd } g \geq 3 . \tag{2.3}$$

The case of even girth leads to the lower bound:

$$n_{[d,g]} \geq 2\frac{(d-1)^{g/2}-1}{d-2} \quad \text{for } d \geq 3 \text{ and even } g \geq 4\ . \tag{2.4}$$

The expressions on the right-hand side of both (2.3) and (2.4) are called the *Moore bounds* for the degree-girth problem.

Finite geometries and generalized k-gons

A *Finite geometry* consists of a finite set of points $\mathcal{P}$ and a finite set of lines $\mathcal{L}$ together with specifications about the points forming a particular line and the lines intersecting in a particular point. Provided that $\mathcal{P}$ and $\mathcal{L}$ are disjoint, this can be completely described by the associated bipartite *point-line incidence graph*, where in any finite bipartite graph one may interpret the two parts in its bipartition as sets of points and lines of a finite geometry. Such a finite bipartite graph of diameter k and girth $2k$ will be called a *generalized k-gon*, see e.g. [19]. A generalized polygon is called *thick* if all its vertices have degree at least 3, and it is known by [18] that a thick generalized k-gon can exist only if $k \in \{3, 4, 6, 8\}$.

If the bipartite graph defining a thick generalized k-gon is assumed to be regular, the order of such a graph is necessarily *equal* to the corresponding Moore bound, thus such graphs will automatically be $[d, 2k]$-cages. For $k = 3$ there exist for any d such that $d-1$ is a prime power, say q, generalized triangles of order $2(d^2 - d + 1) = 2(q^2 + q + 1)$ by (2.4). In the case of $k = 4$ the generalized quadrangles for any d such that $d - 1 = q$ have by (2.4) order $2(d^3 - 2d^2 + 2d) = 2(q^3 + q^2 + q + 1)$.

For the remaining values of k we just note that the situation with regular generalized hexagons (for $k = 6$) is much more complicated, but they also exist under the same condition on d as above, while regular generalized octagons (for $k = 8$) do not exist at all.

The so far best available upper bound for by F. Lazebnik, D. Ustimenko and A. Woldar [23] can be stated as follows: If $d \geq 3$, $g \geq 5$ and q is the smallest odd prime power such that $d \leq q$, then

$$n_{[d,g]} \leq 2dq^{\frac{3}{4}g-\alpha}$$

where $\alpha = 4,\ 11/4,\ 7/2,\ 13/4$ for $g \equiv 0,\ 1,\ 2,\ 3 \bmod 4$, respectively.

Note here that as in the degree-diameter problem, in the degree-girth one the gaps between lower and upper bounds are large. For example, for any fixed odd girth $g \geq 5$ and $d \to \infty$ the lower bound (2.3) is asymptotically $\sim d^{(g-1)/2}$ while the best upper bound of [23] is asymptotically $\sim 2d^{\frac{3}{4}g-c}$ for some constant c.

Vertex-transitive and Cayley of the $[d, g]$-graphs

For $d \geq 3$ and $g = 5$ the only two vertex-transitive cages attaining the (lower) Moore bound are the Petersen graph and the Hoffman-Singleton graph. In the case of $g = 6$ we mention the various constructions of graphs of order 'close' to the Moore bound on $n_{[d,6]}$ for infinitely many degrees d by E. Loz, M. Mačaj, M. Miller, J. Šiagiová, J. Širáň and J. Tomanová [24], that will be relevant for us in Section 4.

Cages of girth $g \geq 5$ attaining the (lower) Moore bound do exist for $g = 6$, 8 and 12, and vertex-transitive ones for $g = 6$ and 8. For larger g, however, the situation is not so favourable, due to an important result of N. L. Biggs [6] showing that for every odd $d \geq 3$ there is an infinite set of values of g such that $n_{[d,g]}$ differs from the (lower) Moore bound for the pair $[d,g]$ by at least g/d.

2.3 Voltage assignments and lifts

In this section we explain the basics about the graph lifting technique using the so-called 'voltage assignments', introduced in [20], which we then apply to constructions of diameter-two lifts of special base graphs in the Section 3.

If Γ is a graph (possibly with loops and parallel edges), then every edge h of Γ can be viewed as consisting of two oppositely directed darts x, x^{-1}, and we write $h = \{x, x^{-1}\}$. Let $V(\Gamma)$ and $D(\Gamma)$ be the vertex set and the dart set of Γ. Given a group G, a *voltage assignment* on Γ in G is a mapping $\alpha : D(\Gamma) \to G$ such that $\alpha(x^{-1}) = (\alpha(x))^{-1}$ for every $x \in D(\Gamma)$. The *lift* Γ^α of Γ by α has vertex set $V(\Gamma^\alpha) = V(\Gamma) \times G$ and dart set $D(\Gamma^\alpha) = D(\Gamma) \times G$, and for any dart x of $D(\Gamma)$ from a vertex u to a vertex v and for any $g \in G$ there is a dart (x, g) in Γ^α from the vertex (u, g) to the vertex $(v, g\alpha(x))$; the darts (x, g) and $(x^{-1}, g\alpha(x))$ form an edge of Γ^α. If G is an Abelian group, then the lift will be called *Abelian*. The original graph Γ is called the *base graph* of the lift.

Given a base graph Γ and a voltage assignment α on Γ in a group G and a walk $W = x_0 x_1 \dots x_t$ in Γ (that is, a sequence of darts such that the terminal vertex of x_{i-1} is the initial vertex of x_i for $i \in \{1, \dots, t\}$), the voltage $\alpha(W)$ is simply the product $\alpha(x_0)\alpha(x_1)\dots\alpha(x_t)$.

The following lemmas 2.1 and 2.2 help us to control diameter and girth in the lift.

Lemma 2.1. *Let α be a voltage assignment on a connected graph Γ in a group G, let k be a positive integer. Then, the lift Γ^α has diameter at most k if and only if for any pair of vertices u, v of Γ, and for every element g of the group G there is a $u \to v$ walk W in Γ of length at most k such that $\alpha(W) = g$.*

Lemma 2.2. [30] *The girth of a lift Γ^α is equal to the length of a shortest closed non-reversing walk W in Γ of net voltage 1_G.*

3 Abelian lifts of graphs

In the paper [29] Šiagiová showed that the McKay-Miller-Širáň graphs presented in Section 2.1 of order (2.2) can be constructed also as lifts of dipoles. Further examination of Abelian lifts of dipoles in the subsequent paper [30] resulted the upper bound in the following Proposition 3.1.

Proposition 3.1. [53] *Let D be a dipole with both vertices of the same degree d and let α be a voltage assignment on D in an Abelian group A, such that the lift D^α has diameter two. Then, the order of D^α is bounded above by*

$$\frac{4(10+\sqrt{2})}{49}(d+0,34)^2 \approx 0,932(d+0,34)^2 \ .$$

This result shows that Abelian lifts of dipoles cannot possibly approach the Moore bound for diameter 2 asymptotically. Nevertheless, Proposition 3.1 still leaves an open possibility to think of beating the the quantity of $\frac{8}{9}\left(d+\frac{1}{2}\right)^2 \approx 0.889d^2$, the orders of the McKay-Miller-Širáň graphs, which motivated our research into Abelian lifts.

Lifting n-poles and (n,n)-bipoles

A natural generalization of dipoles are the complete multigraphs $K_n(m,\ell,s)$ of order $n \geq 2$ and degree $d = (n-1)m + 2\ell + s$, with m parallel edges between any two adjacent vertices, ℓ loops and s semi-edges attached at each vertex. These graphs are regular and we will call them *n-poles*. The second kind of base graphs that generalize dipoles retaining the bipartiteness are complete bipartite multigraphs $K_{n,n}(m,\ell,s)$ for $n \geq 2$, also with edge multiplicity m, ℓ loops and s semi-edges at every vertex; such graphs will be called *(n,n)-bipoles*; they are again regular, of degree $d = mn + 2\ell + s$.

In this section we present preliminary upper bounds on the orders of diameter-two lifts of n-poles and (n,n)-bipoles with voltages in Abelian groups by applying Lemma 2.1.

The following Proposition 3.2 provide the preliminary upper bound on the order of Γ^α as a lift of an n-pole, described as a minimum of two polynomials obtained by examination of the number of distinct voltages on the closed $u \to u$ and $u \to v$ walks of length at most 2 (for distinct $u, v \in V(\Gamma)$) in the base graph, and multiplied by n.

Proposition 3.2. [J4] *Let $\Gamma = K_n(m,\ell,s)$ and let α be a voltage assignment on Γ in an Abelian group A such that the lift Γ^α has diameter two. Then the order of Γ^α is bounded above by*

$$\begin{aligned}\omega_1(m,\ell,s) = n \cdot \min\{&(n-1)m(m-1) + 2\ell(\ell+1) + 2\ell s + \frac{s(s+1)}{2} + 1, \\ &(n-2)m^2 + (4\ell + 2s + 1)m\}.\end{aligned} \tag{3.1}$$

A similar approach leads to the preliminary upper bound on the order of a lift of an (n,n)-bipole, as provided in the next Proposition 3.3.

Proposition 3.3. [J4] *Let $\Gamma = K_{n,n}(m,\ell,s)$ for $n \geq 2$ and let α be a voltage assignment on Γ in an Abelian group A such that the lift Γ^α has diameter two. Then the order of Γ^α is bounded above by*

$$\omega_2(m,\ell,s) = 2n \cdot \min\{nm(m-1) + 2\ell(\ell+1) + 2\ell s + \frac{s(s+1)}{2} + 1, (4\ell + 2s + 1)m, nm^2\}. \quad (3.2)$$

Observe that the 'min' terms in (3.1) and (3.2) also give an upper bounds on the orders of the corresponding Abelian voltage groups.

Now we turn the results stated above in the Propositions 3.2 and 3.3 into more explicit estimates on the order of diameter-two Abelian lifts of an n-pole and an (n,n)-bipole. We begin with n-poles, where the resulting upper bound in terms of d is derived into the following theorem:

Theorem 3.1. [J4] *Let $n \geq 2$ be an integer and let α be a voltage assignment on an n-pole Γ of degree d in an Abelian group such that the lift Γ^α has diameter 2. Then the order of Γ^α is bounded above by*

$$\frac{n^4 + 4n^3 + (2\sqrt{2}-1)n^2 - (2\sqrt{2}+2)n}{(n^2+2n-1)^2} d^2 + O(d^{3/2})$$

as $d \to \infty$.

This result may be regarded as an improvement over the Moore bound for graphs of degree d and diameter two obtained as lifts of n-poles of degree d in an Abelian group. In the case where $n = 2$ we obtain, up to the 'big O' term, the same upper bound as in [30]. For $n = 3$ the upper bound from the Theorem 3.1 gives approximately $\frac{87+6\sqrt{2}}{98} d^2 + O(d^{3/2}) \doteq 0.974 d^2 + O(d^{3/2})$ as $d \to \infty$. This behavior is, of course, expected, since the limit of the leading term in the upper bound in Theorem 3.1 tends to 1 as $n \to \infty$.

In what follows we provide an upper bound on the order of an Abelian diameter-two lift of an (n,n)-bipole:

Theorem 3.2. [J4] *Let $n \geq 2$ be an integer and let α be a voltage assignment in an Abelian group on an (n,n)-bipole Γ of degree d such that the lift Γ^α has diameter 2. If $d \geq 11$, then the order of Γ^α is bounded above by*

$$\frac{8}{9}\left(d + \frac{1}{2}\right)^2 .$$

Theorem 3.2 shows that for every $d \geq 11$ and $n \geq 2$ the upper bound on the order of Abelian lifts of complete bipartite multigraphs based on $K_{n,n}$ of diameter 2 is $\frac{8}{9}\left(d + \frac{1}{2}\right)^2$. Since by (2.2) this bound is equal to the order of McKay-Miller-Širáň graphs [26, 29], the upper bound is sharp. We have found this fact surprising.

The result of the Theorem 3.2 also has the consequence which shows that Abelian lifts of dipoles and (n,n)-bipoles for $n \geq 2$ lead to different upper bounds, and we conclude with mentioning that our upper bound

$$\frac{4(10+\sqrt{2})}{49}(d+0,35)^2 \approx 0,932(d+0,35)^2 ,$$

obtained from lifts of dipoles of odd degrees $d \geq 7$ if both loops as well as semi-edges are allowed, is mildly better than the one presented in the Proposition 3.1.

4 Near-cages of girth 6 from lifts of dipoles

According to the facts about existence of cages of girth 6 and degree d from the Section 2.2 it is of interest to construct families of 'small' $[d, 6]$-graphs also for other sets of degrees. This was considered by Abreu et al. [2], who have shown that for any odd prime power q there exist connected, bipartite graphs of girth 6, degrees $q - i$ for $i \leq 2$, and orders $2q^2$, $2q(q-1)$, $2(q-1)^2$ and $2(q^2-1)$. Later, Loz et al. [24] gave a different construction of these graphs, based on lifting special complete bipartite graphs with voltage assignments in finite fields. In the sense of the conjecture by Pisanski et al. [27] that all g-cages where g is an even integer are bipartite graphs, it is natural to investigate constructions of the previous graphs by lifting from bipartite graphs that are smallest possible, that is, from dipoles.

In the next Section 4.1 we present our results regarding constructions of such graphs based on coverings of graphs by voltage assignments, explained in Section 2.3. In these constructions we also refer to a perfect difference set, that is a set $S \subset C_{q^2+q+1}$ for any prime power q, where for each $a \in C_{q^2+q+1}$, $a \neq 0$ there exists exactly one ordered pair $s, t \in S$ such that $a = s - t$.

4.1 Constructions of small regular graphs of girth 6 by lifting dipoles

Let q be a power of an odd prime and let $F = GF(q)$ be the Galois field of order q. We denote the additive group $(F, +)$ and the multiplicative group $(F \setminus \{0\}, \cdot)$ of F by F^+ and F^*, respectively. Let Γ be the dipole with vertex set $V = \{u, v\}$ and q parallel edges between vertices u and v. We will consider each of those parallel edges to be directed from u to v.

Now we provide our lifting constructions, published in [J3] of graphs from the previous section examined by Abreu et al. [2], and reexamined by Loz et al. [24], arising from a dipole Γ, where most of our voltages will be pairs of elements of F^+ and F^*.

Construction (Γ^α)**.** Define a voltage assignment α on each dart $x \in F$ of a dipole of degree q in the additive group $F^+ \times F^+$ by letting $\alpha(x) = (x, x^2);\ \ x \in F$. The corresponding lift denoted by Γ^α has order $2q^2$.

Construction (Γ^β)**.** Let β be a voltage assignment on each dart $x \in F^*$ of a dipole of degree $q-1$ in the group $F^+ \times F^*$, where $\beta(x) = (x, x);\ \ x \in F^*$. Then the corresponding lift Γ^β has order $2q(q-1)$.

Construction (Γ^γ)**.** Let γ be a voltage assignment on each dart $x \in F^*/\{1\}$ of a dipole of degree $q-2$ in the group $F^* \times F^*$ defined as $\gamma(x) = (x-1, x)$; $x \in F^*/\{1\}$. We denote the resulting lift of order $2(q-1)^2$ by Γ^γ.

Construction (Γ^δ)**.** Let $\tilde{F}$ be an extension field of F, such that $\tilde{F} = GF(q^2) > F$. Define a voltage assignment δ on each dart $x \in \tilde{F}$ of a dipole of degree q by letting $\delta(x) = x + \omega;\ \ x \in F,\ \ \omega \in \tilde{F}^*/F^*,\ \ \omega^2 \in F^*$. Then the resulting lift Γ^δ has order $2(q^2-1)$.

Construction (Γ^ϵ)**.** Let ϵ be a voltage assignment on each dart x; $x \in \{1, 2, \ldots, q+1\}$ of a dipole of degree $q+1$ in the perfect difference set $S \subset C_{q^2+q+1}$; $S = \{s_1, s_2, \ldots, s_{q+1}\}$ defined as $\epsilon(x) = s_x$. Denote the corresponding lift of order $2(q^2+q+1)$ by Γ^ϵ.

It is necessary to prove that the lifts in these constructions have girth $g = 6$, which is stated in the following Proposition 4.1.

Proposition 4.1. [J4] *The lifts Γ^α, Γ^β, Γ^γ, Γ^δ and Γ^ϵ do not contain cycles of length less than or equal to five.*

4.2 Lifting a dipole by a subgroup of a Heisenberg group

In the following Proposition 4.2, we provide a description of another lifting construction of $[q, 6]$-graph Γ^α of order $2q^2$ from the previous section, as a lift of a dipole with voltage assignment from a Heisenberg group.

For any prime power q, the *Heisenberg group* over the Galois field $F = \mathrm{GF}(q)$ is the linear group formed by matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \ a, b, c \in F \tag{4.1}$$

with the usual matrix multiplication as the group operation.

Proposition 4.2. [J5] *Let q be a prime power and let H be the subgroup of the Heisenberg group over $F = \mathrm{GF}(q)$ consisting of the matrices*

$$E(a, b) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}, \ a, b \in F \ .$$

Let Γ be a dipole with vertices u, v joined by q parallel darts $\{e_i, i \in F\}$ indexed by elements of F. Let $\kappa : D(\Gamma) \to H$ be the voltage assignment on the dipole given by

$$\kappa(e_i) = E(i, 0) = \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the lift Γ^κ is a $[q, 6]$-graph of order $2q^2$.

Proposition 4.2 is, in the special case when q is a prime, an extension of the construction given in [9] in terms of the so-called G-graphs.

5 Near-cages of girth 8 and related results

5.1 A family of near-cages of girth 8 from lifting a dipole

In this section we show that the construction of near-cages of girth 6, a prime-power degree q and order $2q^2$ from Proposition 4.2 generalizes for *odd* q to girth 8. The extension is possible due to the existence of a 'favourable' subgroup of a 4-dimensional matrix version of the Heisenberg group.

Let q be an odd prime power and let K_q be the group formed by the following 4-dimensional matrices over F:

$$L(a,b,c) = \begin{pmatrix} 1 & a & a(a-1)/2 & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \ ; \ a,b,c \in F \ .$$

Note that the group K_q has order q^3. Now we provide our result concering the lifts of dipoles with voltages in K_q.

Proposition 5.1. [J5] *For an odd prime power q let K_q be the group introduced above. Let Γ be a dipole with vertices u, v joined by q parallel darts $\{e_i, i \in F\}$ indexed by elements of F. Let $\lambda : D(\Gamma) \to K_q$ be a voltage assignment defined by*

$$\alpha(e_i) = L(i,0,0) = \begin{pmatrix} 1 & i & i(i-1)/2 & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Then the lift Γ^λ is a $[q,8]$-graph of order $2q^3$.

5.2 G-graphs and lifts of dipoles

Another family of graphs we are interested in are called G-graphs, introduced in [7]. Let G be a group and $S = \{s_1, s_2, \ldots, s_n\}$ be a set of elements such that $G = \langle S \rangle$. The *G-graph* $\Phi(G,S)$ is a graph with the vertex set $V(\Phi) = \{\langle s \rangle g; s \in S, g \in G\}$, where $\langle s \rangle g$ is the right coset of the cyclic subgroup $\langle s \rangle$ generated by s, containing g. Two different vertices $\langle s_1 \rangle g$ and $\langle s_2 \rangle h$ are adjacent if and only if the intersection $\langle s_1 \rangle g \cap \langle s_2 \rangle h$ is non-empty.

It follows that $\Phi(G,S)$ is n-partite with $n = |S|$. Let $o(s)$ denote the order of the element s, then since $o(s_1) = o(s_2) = \ldots = o(s_n)$ the graph Φ is regular of degree $o(s)(|S|-1)$ with $o(s)$ loops at each vertex $\langle s \rangle g$. The G-graph with all loops removed will be denoted by $\tilde{\Phi}(G,S)$.

We mention the connection between Cayley graphs and G-graphs, proved in [8]:

Lemma 5.1. *Let $L(\tilde{\Phi}(G,S))$ be the line graph associated with $\tilde{\Phi}$, that is a graph having vertices the edges of $\tilde{\Phi}$ and two vertices in L are adjacent if and only if the corresponding edges in $\tilde{\Phi}$ are incident. Then for generating set $S = \{s_1, s_2\}$ such that $\langle s_1 \rangle \cap \langle s_2 \rangle = \{e\}$ it holds that $L(\tilde{\Phi}(G,S)) \cong X(G,A)$, where $A = (\langle s_1 \rangle \cup \langle s_2 \rangle) \setminus \{e\}$.*

Now we state our result pointing to the relation between G-graphs and lifts of dipoles.

Theorem 5.1. [J5] *Let $G = \langle a, b\rangle$ be a group generated by two elements a and b of the same order k such that $\langle a\rangle \cap \langle b\rangle = \{e\}$. Further, let H be such a subgroup of G for which $H \cap \langle a\rangle = H \cap \langle b\rangle = \{e\}$ and $\langle a\rangle H = \langle b\rangle H = G$. Then the G-graph $\Phi(G, \{a, b\})$ can be described as a lift of dipole with voltage group H and voltage assignment such that $h \in H$ is a voltage on the dart from u to v if and only if $h^{-1} \in \langle a\rangle\langle b\rangle$.*

5.3 An isomorphism from the Heisenberg group onto a Bamberg-Giudici group

Let $p \geq 5$ be a prime number. According to the definition of the Heisenberg group in the Section 4.2, let H_p be the Heisenberg group modulo p, which has representation:

$$H_p = \langle x, y, z \mid x^p = y^p = z^p = 1, xz = zx, yz = zy, zyx = xy\rangle.$$

This group can be also identified with the set of 3×3 matrices

$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in GF(p) \right\}$$

together with classical matrix-multiplication. To see the correspondence between these two representations an arbitrary element of H_p can be expressed in a unique way in the form $z^c y^b x^a$, for suitable generators x, y, z.

In the paper [4] the authors define a group P of order q^3, where q is an odd prime power. In the special case when $q = p$ is an odd prime, the group P (of order p^3) can be described in the form

$$P = \{t_{a,b,0} \cdot \theta_\alpha \mid a, b, \alpha \in GF(p)\},$$

where

$$t_{a,b,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ a & b & 0 & 1 \end{pmatrix} \quad \text{and} \quad \theta_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha & 1 & 0 & 0 \\ -\alpha^2 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{pmatrix},$$

and the group-operation is the classical matrix-multiplication over $GF(p)$.

Now we define a mapping $\varphi : H_p \to P$ as follows:

$$\varphi(x) = \theta_{1/2}, \quad \varphi(y) = t_{0,1,0}, \quad \varphi(z) = t_{1,0,0}.$$

It is easy to check that elements $\varphi(x)$, $\varphi(y)$ and $\varphi(z)$ are elements of order p. Let $\varepsilon = 1/2$ and extend this mapping to an arbitrary element of H_p by letting

$$\varphi(z^c y^b x^a) = \varphi^c(z)\varphi^b(y)\varphi^a(x) = t_{c,0,0} \cdot t_{0,b,0} \cdot \theta_\varepsilon^a = t_{c,b,0} \cdot \theta_\varepsilon^a\ ,$$

where a direct calculation shows that

$$\varphi(z^c y^b x^a) \cdot \varphi(z^w y^v x^u) = \varphi(z^{c+w+av} y^{b+v} x^{a+u}) = \varphi(z^c y^b x^a \cdot z^w y^v x^u).$$

This implies that φ is an isomorphism between the groups H_p and P.

6 Automorphisms of a near-cages of girth 8

In this Chapter we focus on a specific case of near-cages of girth 8 of which we show that they are neither regular lifts of one-vertex graphs (i.e., Cayley graphs) nor regular lifts of dipoles.

The family of $[q,8]$-graphs Υ_q

The family Υ_q of $[q,8]$-graphs are obtained as induced subgraphs of $[q+1,8]$-cage as follows. Let P be a chosen point and ℓ be a chosen line of such a generalized quadrangle. It is known that the vertices in a $[q+1,8]$-cage that are at distance at least 3 from both P and ℓ induce a $[q,8]$-graph of order $2q^3$ if (P,ℓ) is a flag (an incident point-line pair), and of order $2q^3-2q$ if (P,ℓ) is an anti-flag (a non-incident point-line pair).

Now we provide our description of Υ_q by neighbourhoods. Let q be an *odd* prime power. Let us choose the point $P=(\varrho,\varrho,\varrho)_0 \in V(\Gamma_q)$ and the line $\ell=(0,0,0)_1 \in V(\Gamma_q)$. It is easy to check that P is not incident to ℓ, and thus $d_{\Gamma_q}(P,\ell)=3$, since the diameter of Γ_q is 4 and the distance between any point and line is odd. The second closed neighbourhoods are:

$$N^2_{\Gamma_q}[P] = \{(\varrho,\varrho,\varrho)_0,(\varrho,x,y)_0,(\varrho,\varrho,x)_1 : x\in\mathbb{F}_q\cup\{\varrho\}, y\in\mathbb{F}_q\}$$
$$N^2_{\Gamma_q}[\ell] = \{(x,0,0)_0,(\varrho,0,0)_0,(0,x,0)_1,(\varrho,0,x)_1,(\varrho,\varrho,0)_1,(y,-yx,y^2x)_1 : x,y\in\mathbb{F}_q\}.$$

Let Υ_q be the subgraph of Γ_q induced by $V(\Upsilon_q)=V(\Gamma_q)\setminus(N^2_{\Gamma_q}[P]\cup N^2_{\Gamma_q}[\ell])$. Thus, the graph Υ_q contains q^3-q points and q^3-q lines corresponding to V_0 and V_1, respectively:

$$V_0=\{(x,y,z)_0, x,y,z\in\mathbb{F}_q\}\setminus\{(x,0,0)_0 : x\in\mathbb{F}_q\},$$
$$V_1=\{(\varrho,j,k)_1 : j,k\in\mathbb{F}_q, j\neq 0\}\cup\{(i,j,k)_1 : i,j,k\in\mathbb{F}_q, ij+k\neq 0\}.$$

The edge set of Υ_q can be described with the help of neighbourhoods as follows:

$$N_{\Upsilon_q}((x,0,z)_0)=\{(r,-rx,r^2x+z)_1, r\in\mathbb{F}_q\},\ \text{for all } x,z\in\mathbb{F}_q,\ z\neq 0;$$
$$N_{\Upsilon_q}((x,y,z)_0)=\{(r,y-rx,r^2x-2ry+z)_1, r\in\mathbb{F}_q, r\neq y^{-1}z\}\cup$$
$$\cup\{(\varrho,y,x)_1\},\ \text{for all } x,y,z\in\mathbb{F}_q,\ y\neq 0.$$

Thus, the graph Υ_q is regular of valency q.

Further, let us define the following subsets of vertices of Υ_q.

$$O_1=\{(x,y,0)_0 : x,y\in\mathbb{F}_q, y\neq 0\}$$
$$O_2=\{(x,y,z)_0 : x,y,z\in\mathbb{F}_q, z\neq 0\}$$
$$O_3=\{(0,j,k)_1 : j,k\in\mathbb{F}_q, k\neq 0\}$$
$$O_4=\{(i,j,k)_1 : i,j,k\in\mathbb{F}_q, i\neq 0, ij+k\neq 0\}\cup\{(\varrho,j,k)_1, j,k\in\mathbb{F}_q, j\neq 0\}.$$

Clearly, V_0 is a disjoint union of O_1 and O_2, while V_1 is a disjoint union of O_3 and O_4.

The basic facts about these sets can be summed up as follows.

Corollary 6.1. [J6] *There is no automorphism of Υ_q mapping a vertex from $O_1\cup O_2$ to a vertex in $O_3\cup O_4$, and vice versa.*

Corollary 6.2. [J6] *Let $q > 3$ be an odd prime power. Then the diameter of Υ_q is equal to 6.*

Corollary 6.3. [J6] *Let G be the full group of automorphisms of the graph Υ_q. Then G has at least four orbits on $V(\Upsilon_q)$.*

From Corollary 6.3 the main result of this section follows as a consequence.

Theorem 6.1. [J6] *The graph Υ_q cannot be obtained as a Cayley graph, nor as a lift of a dipole.*

Since the group of automorphisms of Υ_q has at least four orbits, none of its subgroups can act regularly, or semi-regularly with two orbits on the set of vertices, therefore Υ_q is neither a Cayley graph, nor a lift of a dipole.

Automorphisms of the graph Υ_q

In this section we describe several mappings on $V(\Upsilon_q)$ and we show that they all are automorphisms of the graph Υ_q.

For all $\alpha, \beta \in \mathbb{F}_q$, $\alpha \neq 0$, let us define the mapping $\varphi_{\alpha,\beta} : V(\Upsilon_q) \to V(\Upsilon_q)$ as follows:

$$\begin{aligned}
\varphi_{\alpha,\beta}((x,y,z)_0) &= (\alpha x + \beta, \alpha y, \alpha z)_0, \\
\varphi_{\alpha,\beta}((i,j,k)_1) &= (i, \alpha j - \beta i, \alpha k + \beta i^2)_1 \\
\varphi_{\alpha,\beta}((\varrho,j,k)_1) &= (\varrho, \alpha j, \alpha k + \beta)_1.
\end{aligned}$$

By noticing that $\varphi_{\alpha,\beta}$ is closed on V_0 and on V_1, a routine computations show that $\varphi_{\alpha,\beta}$ is a bijective mapping on $V(\Upsilon_q)$, and that the set

$$H = \{\varphi_{\alpha,\beta} : \alpha, \beta \in \mathbb{F}_q, \alpha \neq 0\}$$

is closed under the operation $\circ$ of composition of mappings. We conclude that $(H, \circ)$ is a group isomorphic to the affine linear group $AGL_1(q)$.

In Υ_q there are just two types of edges. Consider first an edge e of type $\{(i,j,k)_1, (x, ix + j, i^2x + 2ij + k)_0\}$, where $i,j,k,x \in \mathbb{F}_p$ and $ij + k \neq 0$, which image under $\varphi_{\alpha,\beta}$ is an edge in Υ_q. On the other hand, an edge of type $\{(\varrho,j,k)_1, (k,j,x)_0\}$, where $j,k,x \in \mathbb{F}_q$ and $j \neq 0$ is mapped to the pair $\{(\varrho, \alpha j, \alpha k + \beta)_1, (\alpha k + \beta, \alpha j, \alpha x)_0\}$ which, again, is an edge of Υ_q. Thus, $H \leq \mathrm{Aut}(\Upsilon_q)$.

For all $\alpha, \beta \in \mathbb{F}_q$, $\alpha \neq 0$, we define the mapping $\sigma_{\alpha,\beta} : V_0 \to V_0$ as follows:

$$\sigma_{\alpha,\beta}((x,y,z)_0) = \left(x + \frac{2\beta}{\alpha}y + \frac{\beta^2}{\alpha^2}z, \alpha y + \beta z, \alpha^2 z\right)_0.$$

After showing that the mapping $\sigma_{\alpha,\beta}$ is a bijection on the set V_0 as well as V_1, we prove that the set

$$K = \{\sigma_{\alpha,\beta} : \alpha, \beta \in \mathbb{F}_q, \alpha \neq 0\}$$

is a set of automorphism of Υ_q.

For the last kind of a mapping we recall that the full group of automorphisms of a finite field of order $q = p^n$ for any prime p is known to consist of the so-called *Frobenius automorphisms*, given, for any fixed $r \in \{0, 1, \ldots, n-1\}$, by $\pi_r : \mathbb{F}_q \to \mathbb{F}_q$, $x \mapsto x^{p^r}$ for each $x \in \mathbb{F}_q$.

We can extend each such automorphism by defining $\pi_r : \varrho \mapsto \varrho$, which induces naturally (component-wise on the coordinates) a mapping on the vertices of the graph Υ_q. It is not hard to check that these Frobenius automorphisms generate automorphisms of the graph Υ_q.

Let H and K be groups obtained above, that have trivial intersection. For all $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$, $\alpha, \gamma \neq 0$ we have that

$$\sigma_{\alpha,\beta} \circ \varphi_{\gamma,\delta} = \varphi_{\gamma,\delta} \circ \sigma_{\alpha,\beta}.$$

In other words, elements of H commute with the elements of K, thus we have the following lemma.

Lemma 6.1. [J6] *Let H and K be the groups defined above. Then $H \times K \leq Aut(\Upsilon_q)$, or equivalently, the full group of automorphisms of Υ_q contains a subgroup isomorphic to $AGL_1(q) \times AGL_1(q)$.*

If we consider the automorphisms of Υ_q generated by the Frobenius automorphisms of the finite field $\mathbb{F}_q$, we can extend the latter lemma.

Lemma 6.2. [J6] *Let $\mathbb{F}_q$ be a finite field of order $q = p^n$, where p is an odd prime and n an integer. Then*

$$\mathrm{Aut}(\Upsilon_q) \geq (H \times K) \rtimes \mathcal{F} \cong (AGL_1(q) \times AGL_1(q)) \rtimes \mathbb{Z}_n.$$

This all finally leads to the following result.

Theorem 6.2. [J6] *Let $G = \mathrm{Aut}(\Upsilon_q)$ be the (full) group of automorphisms of Υ_q. Then G has exactly four orbits on $V(\Upsilon_q)$ and these orbits are precisely the sets O_1, O_2, O_3 and O_4. Two of the orbits have length $q(q-1)$ and the remaining two have length $q^2(q-1)$.*

7 Conclusion and future work

Conclusion of our results

In this dissertation we have focused on lifting constructions in the degree-diameter problem and degree-girth problem.

In Section 3 we have presented new results regarding upper bounds on diameter-two Abelian lifts of relatively small base graphs, namely dipoles, n-poles and (n, n)-bipoles. Our results have pointed out that the order of the currently largest Abelian lifts of dipoles (the McKay-Miller-Širáň graphs) cannot be further improved.

Section 4 contains five known families of 'near-cages' of a given degree and girth 6, all of them constructed as Abelian lifts of dipoles over the Galois field of order q, a power of an odd prime. For the first of these constructions, denoted by Γ^α, we have given a new alternative lifting construction Γ^κ with the suitable voltage assignments from the Heisenberg group.

We also have produced a family of known 'near-cages' of girth 8 as an extension of the Γ^κ construction, motivated by the so-called G-graphs (Section 5). At the end of this section we have explained links between a special case of G-graph construction and the lifting construction and revealed a related isomorphism of two matrix groups studied in this context.

Finally, in the Section 6 we have showed by detailed investigation of the automorphism groups that graphs in another well known family of 'near-cages' of girth 8, provided in [1], are neither regular lifts of one-vertex graphs (i.e., Cayley graphs) nor regular lifts of dipoles.

Future work

The development in the study of lifting constructions in the degree-diameter and the degree-girth problem, as well as examining the groups used for constructing a family of Cayley graphs of a given degree and diameter 3 in [3], clearly demonstrates that further progress is unlikely without involving more complicated voltage groups. Thus one of the possible directions for further research in this area could be to investigate more complicated voltage groups used on the dipole, which could also lead to constructions of bi-regular graphs with given properties.

Another possible approach of research in the degree-diameter and the degree-girth problem, stemming from our results in Section 6, is to shift the focus on lifts of base graphs with a larger number of vertices, similar to the original approach in the construction of the McKay-Miller-Širáň graphs in [25].

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- [J1] **P. Jánoš**, Lifting complete bipartite multigraphs to graphs of diameter two, Advances in Architectural, Civil and Environmental Engineering. Bratislava: Spektrum STU, 2018, s. 20-26. ISBN 978-80-227-4864-3.

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- [J6] Š. Gyürki and **P. Jánoš**, On the automorphisms of a family of small q-regular graphs of girth 8, submitted for publication in Art of Discrete and Applied Mathematics.