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Abstract

In this dissertation thesis, we obtained multiple results to broaden the theory of aggregation functions. In particular, we have turned our attention to the class of decomposition integrals, a general framework containing many non-linear integrals, including the Choquet integral, the Shilkret integral, the PAN integral, and the concave integral. Decomposition integrals extend the idea of lower Riemann sums which allows one to consider not only measures, but a more general set functions called monotone measures. An analogous notion of Riemann upper sums leads to the definition of super-decomposition integrals. Firstly, we have investigated a particular sub-class of decomposition integrals for singleton decomposition systems and introduced so-called collection integrals. With regards to collection integrals, we investigated integral inequalities and characterized all collection integrals representable by the Choquet integral. We have also introduced a generalization of a collection integral, called a super-additive integral, applicable in the theory of imprecise probabilities, particularly in constructions of some new types of lower/upper coherent previsions. For decomposition integrals, some results have been obtained for an open problem of their coincidence with the Lebesgue integral. We have introduced a few new transformations for aggregation functions, notably a k-bounded transformation and a revenue transformation. Also, the convolution of aggregation functions has been introduced and investigated in the setting of decomposition integrals. After applying super- and sub-additive transformations, we have obtained some results on continuity preservation of aggregation functions.

keywords: decomposition integral · collection integral · integral inequality · aggregation function · transformation of aggregation functions · coherent lower prevision · coherent upper prevision · extension of Lebesgue integral

Abstrakt

V tejto dizertačnej práci sú zhrnuté viaceré výsledky, ktoré sme dosiahli v teórii agregačných funkcií. Zamerali sme sa najmä na triedu dekompozičných integrálov, ktoré sú všeobecným rámcom zahŕňajúcim mnohé nelineárne integrály, vrátane Choquetovho integrálu, Shilkretovho integrálu, PAN integrálu a konkávneho integrálu. Dekompozičné integrály sú založené na myšlienke Riemannovych dolných súm, čo umožňuje použiť nielen miery, ale aj všeobecnejšie množinové funkcie nazývané monotónne miery. Analogická myšlienka Riemannovych horných súm vedie k definícii super-dekompozičných integrálov. V prvom rade sme sa zamerali na výskum špeciálnej podtriedy dekompozičných integrálov vzhľadom na jednoprvkové dekompozičné systémy, čo viedlo k zavedeniu tzv. integrálov na kolekciách. Pre integrály na kolekciách sme skúmali integrálne nerovnosti a charakterizovali sme všetky integrály na kolekciách, ktoré sú reprezentovateľné Choquetovým integrálom. Takisto sme zaviedli rozšírenie integrálu na kolekciách, ktoré sme nazvali super-aditívnym integrálom, pričom toto rozšírenie našlo svoje uplatnenie v teórii nepresných pravdepodobností, špeciálne v konštrukciách dolných/horných koherentných prevízií. Pri skúmaní dekompozičných integrálov sme získali niekoľko výsledkov pri snahe vyriešiť otvorený problém charakterizácie tých dekompozičných integrálov, ktoré rozširujú Lebesgueov integrál. V teórii agregačných funkcií sme zaviedli niektoré nové transformácie, predovšetkým k-ohraničené transformácie a transformáciu návratu. Zároveň sme zaviedli konvolúciu agregačných funkcií a skúmali jej vplyv na dekompozičné integrály. Potom sme skúmali zachovanie spojitosti agregačných funkcií po aplikovaní super- a sub-aditívnych transformácií zavedených Grecom a spol.

kľúčové slová: dekompozičný integrál · integrál na kolekciách · integrálna nerovnosť · agregačná funkcia · transformácia agregačných funkcií · dolná koherentná prevízia · horná koherentná prevízia · rozšírenie Lebesgueovho integrálu

Contents

Chapter 1

Introduction

The theory of integration has a long history beginning with a simple idea of computing areas under a function curve. If we consider any (well-behaving) function curve, we can approximate the area under the curve using simpler geometric shapes whose area is known. The first historically documented method to do this was Greek astronomer Eudoxus' Method of Exhaustion. Modern foundations of the theory of integration were given by Cavalieri and Fermat in the 17^{th} century. Later, two independent mathematicians, namely Leibniz and Newton, found a link connecting the theory of integration with the theory of differentiation. Unfortunately, even though Leibniz and Newton provided a systematic integration approach, it lacked a rigorous background. The first rigorous approach to integration was given by Riemann and, later, a different integration approach by Lebesgue.

However, both the Riemann and Lebesgue integrals fail to model subjective reasoning, which occurs in many real-life situations. Choquet, in his research work, proposed a new integral, which integrated functions with respect to monotone measures, generalizations of classical measures, and allowed modelling of subjective and non-linear reasoning. Since then, many different types of non-linear integrals have been proposed, notably the Sugeno, the Shilkret, the PAN, and the concave integrals. In 2014, Even and Lehrer's brilliant idea allowed them to define a framework of non-linear integrals, called decomposition integrals, which include all of the above-mentioned non-linear integrals.

Our research focused on the theory of decomposition integrals, specifically as follows: to examine general and individual properties of decomposition integrals; to introduce new types of decomposition integrals; to find methods of computing decomposition integrals; to find the decomposition integrals that extend the Lebesgue integral; to define decomposition integrals on more general spaces; to find applications of decomposition integrals; and to find new non-linear integrals.

Since decomposition integrals are a special kind of continuous aggregation functions, we also turned our attention to them. In the theory of aggregation functions, we started to solve the problem of continuity preservation after applying the super- and sub-additive transformations of aggregation functions and we obtained many interesting results. Then, we defined some new transformations of aggregation functions, and also introduced four different ways to convolute them.

This thesis is organized as follows: the upcoming section covers basic definitions of and results on aggregation functions in general, their properties, and their super- and sub-additive transformations. This is followed by the definition of a few non-linear integrals and the framework of decomposition integrals. Chapter 3 summarizes obtained results on decomposition integrals and includes the following sections: (i) results and construction methods of decomposition integrals extending the Lebesgue integral; (ii) extension of decomposition integrals for interval-valued functions; (iii) results and applications of decomposition integrals with respect to singleton decomposition integrals; (iv) modification of decomposition integrals based on the Knapsack problem; and (v) a discussion on the computational complexity of computing specific decomposition integrals. Chapter 4 consists of the results on aggregation functions and is divided into the following two parts: (i) definitions of new transformations of aggregation functions and the problem of continuity preservation after applying the super- and sub-additive transformations of aggregation functions; and (ii) methods of convoluting the aggregation functions. Chapter 5 contains concluding remarks.

Chapter 2

Aggregation functions

Aggregation functions [1, 7] are functions describing the process of aggregation. These functions obey the following: if you have minimal (maximal) possible inputs, you should obtain the minimal (maximal) possible output. If you increase your input, your output should not decrease.

In the previous description, we did not characterize what inputs and output we will consider. As explained above, by inputs, we understand an n-tuple of some values, and by an output, we understand a single value, all from the same scale. In what follows, we restrict our considerations to the scales $[0, \infty[$ and $[0, 1]$.

Definition 2.1. Let $n \in \mathbb{N}$ be a natural number. An *aggregation function* is any mapping $\mathsf{A}\colon [0, \infty[^n \to [0, \infty[$ such that the boundary condition $\mathsf{A}(\mathbf{0}) = 0$ is satisfied, and if $\mathbf{x} \leq \mathbf{y}$ then $\mathsf{A}(\mathbf{x}) \leq \mathsf{A}(\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in [0, \infty[^n$.

Example 2.1. The following three mappings $\mathsf{F}, \mathsf{G}, \mathsf{H}\colon [0, \infty[\to [0, \infty[$ given by

$$\mathsf{F}(x) = \min\{x, 1\}, \quad \mathsf{G}(x) = \sqrt{x} \quad \text{and} \quad \mathsf{H}(x) = x^2$$

are all one-dimensional aggregation functions.

Example 2.2. As an example of a multiple-dimensional aggregation function, we can consider the l_2 norm of a vector, i.e., an aggregation function

$$\mathsf{K}\colon [0, \infty[^n \to [0, \infty[\quad \text{such that} \quad \mathsf{K}(\mathbf{x}) = \|\mathbf{x}\|\,,$$

e.g., for $n = 2$ we have $\mathsf{K}(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$. Another example is the product of all the coordinates of a vector, i.e., an aggregation function

$$\mathsf{L}\colon [0, \infty[^n \to [0, \infty[\quad \text{such that} \quad \mathsf{L}(\mathbf{x}) = \prod_{i=1}^{n} x_i,$$

e.g., for $n = 2$ we obtain $\mathsf{L}(\mathbf{x}) = x_1 x_2$.

Definition 2.2. Let $n \in \mathbb{N}$ be a natural number. A $[0, 1]$-*aggregation function* is any mapping $\mathsf{A}\colon [0, 1]^n \to [0, 1]$ such that boundary conditions $\mathsf{A}(\mathbf{0}) = 0$ and $\mathsf{A}(\mathbf{1}) = 1$ hold, and if $\mathbf{x} \leq \mathbf{y}$ then $\mathsf{A}(\mathbf{x}) \leq \mathsf{A}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n$.

Different properties of aggregation functions and $[0, 1]$-aggregation functions can be under consideration. In the following, we summarize some of the properties. Analogous definitions can be given for $[0, 1]$-aggregation functions.

Definition 2.3. An aggregation function $\mathsf{A} \in \mathbb{A}$ is said to be

- *homogeneous*, if $\mathsf{A}(\alpha \mathbf{x}) = \alpha \mathsf{A}(\mathbf{x})$ for all $\alpha \in [0, \infty[$ and $\mathbf{x} \in [0, \infty[^n$;
- *shift-invariant*, if $\mathsf{A}(\mathbf{x} + \alpha \mathbf{1}) = \mathsf{A}(\mathbf{x}) + \alpha \mathsf{A}(\mathbf{1})$ for all $\alpha \in [0, \infty[$ and $\mathbf{x} \in [0, \infty[^n$;
- *super-additive*, if $\mathsf{A}(\mathbf{x} + \mathbf{y}) \geq \mathsf{A}(\mathbf{x}) + \mathsf{A}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in [0, \infty[^n$;

- *sub-additive*, if $\mathsf{A}(\mathbf{x}+\mathbf{y}) \le \mathsf{A}(\mathbf{x}) + \mathsf{A}(\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in [0,\infty[^n$.

Now, we can begin a discussion about super- and sub-additive transformations of aggregation functions [8]. The super-additive transformation exists only for some aggregation functions. Their characterization is in the following definition.

Definition 2.4. We say that an aggregation function $\mathsf{A} \in \mathbb{A}$ *escapes* if there exists $\mathbf{x} > \mathbf{0}$ such that

$$\bigvee\left\{\sum_{i=1}^{k}\mathsf{A}(\mathbf{x}_i) : \sum_{i=1}^{k}\mathbf{x}_i = \mathbf{x}, \mathbf{x}_i \ge 0 \text{ for all } i = 1,2,\dots,k;\ k \in \mathbb{N}\right\} = \infty$$

and *does not escape* if for all $\mathbf{x} > \mathbf{0}$ one has

$$\bigvee\left\{\sum_{i=1}^{k}\mathsf{A}(\mathbf{x}_i) : \sum_{i=1}^{k}\mathbf{x}_i = \mathbf{x}, \mathbf{x}_i \ge 0 \text{ for all } i = 1,2,\dots,k;\ k \in \mathbb{N}\right\} < \infty.$$

The class of all aggregation functions that do not escape is denoted by $\mathbb{A}^+$.

Now, let us introduce super- and sub-additive transformations of aggregation functions.

Definition 2.5. Let $\mathsf{A}\colon [0,\infty[^n \to [0,\infty[$ be an aggregation function such that $\mathsf{A} \in \mathbb{A}^+$. Its *super-additive transformation* is an aggregation function $\mathsf{A}^*\colon [0,\infty[^n \to [0,\infty[$ given by

$$\mathsf{A}^*(\mathbf{x}) = \bigvee\left\{\sum_{i=1}^{k}\mathsf{A}(\mathbf{x}_i) : \sum_{i=1}^{k}\mathbf{x}_i = \mathbf{x}, \mathbf{x}_i \ge 0 \text{ for all } i = 1,2,\dots,k;\ k \in \mathbb{N}\right\}$$

for all $\mathbf{x} \in [0,\infty[^n$.

Definition 2.6. Let $\mathsf{A}\colon [0,\infty[^n \to [0,\infty[$ be an aggregation function. Its *sub-additive transformation* is an aggregation function $\mathsf{A}_*\colon [0,\infty[^n \to [0,\infty[$ given by

$$\mathsf{A}^*(\mathbf{x}) = \bigwedge\left\{\sum_{i=1}^{k}\mathsf{A}(\mathbf{x}_i) : \sum_{i=1}^{k}\mathbf{x}_i = \mathbf{x}, \mathbf{x}_i \ge 0 \text{ for all } i = 1,2,\dots,k;\ k \in \mathbb{N}\right\}$$

for all $\mathbf{x} \in [0,\infty[^n$.

Example 2.3. Let $\mathbf{x}$ denote a vector of available resources and let $\mathsf{A}(\mathbf{x})$ denote the profit from selling these resources as a whole. Then $\mathsf{A}^*(\mathbf{x})$ denotes the supremal profit obtainable by dividing our resources to k groups $\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_k$. On the other hand, if $\mathbf{x}$ denotes a vector of lacking resources and $\mathsf{A}(\mathbf{x})$ is the price at which we can buy these resources, the sub-additive transformation evaluated at point $\mathbf{x}$, i.e., $\mathsf{A}_*(\mathbf{x})$, is the infimal price for $\mathbf{x}$ resources which can be bought by dividing them into smaller groups $\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_k$, $k \in \mathbb{N}$.

Also, in the paper [8], some properties of the super- and sub-additive transformations were given. They are summarized in the following proposition.

Proposition 2.1. Let $\mathsf{A},\mathsf{B} \in \mathbb{A}$ be two aggregation functions (for the super-additive transformation, assume $\mathsf{A},\mathsf{B} \in \mathbb{A}^+$). Then

- $\mathsf{A}^* \ge \mathsf{A} \ge \mathsf{A}_*$;
- A is super-additive (sub-additive) if and only if $\mathsf{A}^* = \mathsf{A}$ ($\mathsf{A}_* = \mathsf{A}$);
- $\mathsf{A}^{**} = \mathsf{A}^*$ and $\mathsf{A}_{**} = \mathsf{A}_*$; and
- if $\mathsf{A} \le \mathsf{B}$ then $\mathsf{A}^* \le \mathsf{B}^*$ and $\mathsf{A}_* \le \mathsf{B}_*$.

Example 2.4. For one-dimensional aggregation functions $\mathsf{F},\mathsf{G},\mathsf{H}$ introduced in Example 2.1 one obtains

$$\mathsf{F}^*(x) = x, \quad \mathsf{G} \text{ escapes} \quad \text{and} \quad \mathsf{H}^*(x) = x^2,$$

and

$$\mathsf{F}_*(x) = \min\{x,1\}, \quad \mathsf{G}_*(x) = \sqrt{x} \quad \text{and} \quad \mathsf{H}_*(x) = 0.$$

Typically, integration takes place over some space X and we will only consider a finite space X, where we can assume that without loss of any generality, $X = \{1, 2, \dots, n\}$ for some natural number $n \in \mathbb{N}$.

Definition 2.7. A *monotone measure* is any set function $\mu: 2^X \to [0, \infty[$ such that μ is grounded, i.e., $\mu(\emptyset) = 0$, and μ is non-decreasing with respect to set inclusion, i.e., $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ for all sets $A, B \subseteq X$. A class of all monotone measures will be denoted by a symbol $\mathbb{M}$.

Similar to aggregation functions, monotone measures can be labelled by different properties.

Definition 2.8. A monotone measure $\mu \in \mathbb{M}$ is said to be

- *additive*, if $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in 2^X$ such that $A \cap B = \emptyset$;
- *super-additive*, if $\mu(A \cup B) \geq \mu(A) + \mu(B)$ for all $A, B \in 2^X$ such that $A \cap B = \emptyset$;
- *sub-additive*, if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for all $A, B \in 2^X$ such that $A \cap B = \emptyset$.

An additive monotone measure is typically referred to as a *measure*. A class of all measures will be denoted by $\mathbb{M}_+$.

Remark 2.1. To simplify the notation we will adopt the following: if $\mathbf{x}$ is an n-dimensional vector and $A \subseteq X$, $A \neq \emptyset$, then $\mathbf{x}_A$ is $|A|$-dimensional vector with values selected from the original vector $\mathbf{x}$ on coordinates with indices in A. By $\bigvee \mathbf{x}_A$ we understand the maximal value in the coordinates of the vector $\mathbf{x}_A$ and, analogously, by $\bigwedge \mathbf{x}_A$ the minimal value. If vector $\mathbf{x} \in \mathbb{F}$ represents a function $f: X \to [0, \infty[$, then $\mathbf{x}_A$ is equivalent to $f(A)$, $\bigvee \mathbf{x}_A$ is equivalent to $\max f(A)$ and $\bigwedge \mathbf{x}_A$ is equivalent to $\min f(A)$. By convention, we will assume that $\bigvee \mathbf{x}_\emptyset = 0$, $\bigwedge \mathbf{x}_\emptyset = \infty$ and $\infty \cdot 0 = 0 = 0 \cdot \infty$.

Example 2.5. In this example, some integrals $\mathbb{F} \times \mathbb{M} \to [0, \infty[$ are summarized:

- *Lebesgue integral* [10]
$$\mathsf{Leb}(\mathbf{x}, \mu) = \sum_{i=1}^{n} x_i \mu(\{i\});$$
- *Choquet integral* [2]
$$\mathsf{Ch}(\mathbf{x}, \mu) = \sum_{i=1}^{n} \big(x_{(i)} - x_{(i-1)}\big) \mu(A_{(i)}),$$
where $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ is a non-decreasing enumeration of coordinates of $\mathbf{x}$ and $A_{(i)} = \{i \in X: x_i \geq x_{(i)}\}$ for $i = 1, 2, \dots, n$ with a convention $x_{(0)} = 0$;
- *Shilkret integral* [28]
$$\mathsf{Sh}(\mathbf{x}, \mu) = \bigvee_{A \subseteq X} \Big(\mu(A) \cdot \bigwedge \mathbf{x}_A\Big);$$
- *the PAN integral*
$$\mathsf{PAN}(\mathbf{x}, \mu) = \bigvee \left\{ \sum_{i=1}^{k} \Big(\mu(A_i) \cdot \bigwedge \mathbf{x}_{A_i}\Big) : \{A_1, A_2, \dots, A_k\} \text{ is a partition of } X \right\};$$
- *the concave integral* [11]
$$\mathsf{con}(\mathbf{x}, \mu) = \bigvee \left\{ \sum_{A \subseteq X} \alpha_A \mu(A) : \sum_{A \subseteq X} \alpha_A \mathbb{1}_A \leq \mathbf{x}, \alpha_A \geq 0 \text{ for all } A \subseteq X \right\};$$
- *the convex integral*
$$\mathsf{cvx}(\mathbf{x}, \mu) = \bigwedge \left\{ \sum_{A \subseteq X} \alpha_A \mu(A) : \sum_{A \subseteq X} \alpha_A \mathbb{1}_A \geq \mathbf{x}, \alpha_A \geq 0 \text{ for all } A \subseteq X \right\}.$$

Example 2.6. For these integrals, see [8], one has $\mathsf{Sh}^* = \mathsf{Ch}^* = \mathsf{con} = \mathsf{con}^*$ and $\mathsf{Ch}_* = \mathsf{cvx} = \mathsf{cvx}_*$. Also, for any aggregation function A such that $\mathsf{Sh} \leq \mathsf{A} \leq \mathsf{con}$ one obtains $\mathsf{A}^* = \mathsf{con}$.

In [5], the framework of *decomposition integrals* was introduced.

Definition 2.9. A non-empty finite subset of $2^X \setminus \{\varnothing\}$ is called a *collection* and the class of all collections will be denoted by a symbol $\mathbb{D}$. A non-empty set of collections is called a *decomposition system* and the class of all decomposition systems will be denoted by a symbol $\mathbb{H}$.

Definition 2.10. Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system. An operator $\mathsf{dec}_{\mathcal{H}}\colon \mathbb{F} \times \mathbb{M} \to [0,\infty[$ given by

$$\mathsf{dec}_{\mathcal{H}}(f,\mu) = \bigvee_{\mathcal{D}\in\mathcal{H}} \bigvee \left\{ \sum_{A\in\mathcal{D}} \alpha_A \mu(A) \colon \sum_{A\in\mathcal{D}} \alpha_A \mathbb{1}_A \le f, \alpha_A \ge 0 \text{ for all } A \in \mathcal{D} \right\}$$

for all functions $f \in \mathbb{F}$ and all monotone measures $\mu \in \mathbb{M}$ is called a *decomposition integral* with respect to the decomposition system $\mathcal{H}$. A decomposition integral with respect to a singleton decomposition system $\mathcal{H} = \{\mathcal{D}\}$, where $\mathcal{D} \in \mathbb{D}$, is called a *collection integral* [19] with respect to a collection $\mathcal{D}$ and will be denoted by $\mathsf{col}_{\mathcal{D}}$.

Example 2.7. In the framework of decomposition integrals many integrals that are heavily used can be found:

- if $\mathcal{H}_{\mathsf{Sh}} = \big\{\{A\} \colon A \in 2^X \setminus \{\varnothing\}\big\}$ then $\mathsf{dec}_{\mathcal{H}_{\mathsf{Sh}}}$ is the Shilkret integral;
- if $\mathcal{H}_{\mathsf{PAN}} = \big\{\mathcal{D} \in \mathbb{D} \colon \mathcal{D} \text{ is a partition of } X\big\}$ then $\mathsf{dec}_{\mathcal{H}_{\mathsf{PAN}}}$ is the PAN integral;
- if $\mathcal{H}_{\mathsf{Ch}} = \big\{\mathcal{D} \in \mathbb{D} \colon \mathcal{D} \text{ is a chain in } 2^X \setminus \{\varnothing\}\big\}$ then $\mathsf{dec}_{\mathcal{H}_{\mathsf{Ch}}}$ is the Choquet integral;
- if $\mathcal{H}_{\mathsf{con}} = \big\{2^X \setminus \{\varnothing\}\big\}$ then $\mathsf{dec}_{\mathcal{H}_{\mathsf{con}}}$ is the concave integral; and
- if $\mathcal{H}_{\mathsf{PC}} = \big\{\mathcal{D} \in \mathbb{D} \colon \text{if } A, B \in \mathcal{D} \text{ then } A \cap B \in \{A, B, \varnothing\}\big\}$ then $\mathsf{dec}_{\mathcal{H}_{\mathsf{PC}}}$ is the PC integral [30].

There is an analogy between decomposition integrals and lower Riemann integral sums. Based on the idea of upper Riemann integral sums, super-decomposition integrals were introduced in [13].

Definition 2.11. Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system. An operator $\mathsf{dec}^*_{\mathcal{H}}\colon \mathbb{F} \times \mathbb{M} \to [0,\infty]$ given by

$$\mathsf{dec}^*_{\mathcal{H}}(f,\mu) = \bigwedge_{\mathcal{D}\in\mathcal{H}} \bigwedge \left\{ \sum_{A\in\mathcal{D}} \alpha_A \mu(A) \colon \sum_{A\in\mathcal{D}} \alpha_A \mathbb{1}_A \ge f, \alpha_A \ge 0 \text{ for all } A \in \mathcal{D} \right\}$$

for all functions $f \in \mathbb{F}$ and all monotone measures $\mu \in \mathbb{M}$ is called a *super-decomposition integral* with respect to the decomposition system $\mathcal{H}$. A super-decomposition integral with respect to a singleton decomposition system $\mathcal{H} = \{\mathcal{D}\}$, where $\mathcal{D} \in \mathbb{D}$, is called a *super-collection integral* with respect to a collection $\mathcal{D}$ and will be denoted by $\mathsf{col}^*_{\mathcal{D}}$.

Example 2.8.

- the operator $\mathsf{dec}^*_{\mathcal{H}_{\mathsf{con}}}$ is the convex integral; and
- the operator $\mathsf{dec}^*_{\mathcal{H}_{\mathsf{Ch}}}$ is the Choquet integral.

Other sub-classes of aggregation functions, e.g., copulas, will be described later as necessary.

Chapter 3

Results on decomposition integrals

3.1 Decomposition integrals extending Lebesgue integral

In the paper [12], we examined the decomposition integrals that extend the Lebesgue integral. The main objective of the paper was to characterize the class

$$\mathbb{H}_+ = \big\{\mathcal{H} \in \mathbb{H} : \mathsf{dec}_{\mathcal{H}} \text{ extends } \mathsf{Leb}\big\},$$

i.e., the decomposition systems for which the corresponding decomposition integral extends the Lebesgue integral, $\mathsf{dec}_{\mathcal{H}}(f,\mu) = \mathsf{Leb}(f,\mu)$ for all $f \in \mathbb{F}$ and $\mu \in \mathbb{M}_+$.

The basic characterization of decomposition systems $\mathcal{H} \in \mathbb{H}$ such that $\mathsf{dec}_{\mathcal{H}}$ coincides with Leb if we only restrict ourselves to the measures given in the original paper [5]. To simplify the notation, we introduce the $\mathcal{D}$-decomposability:

Definition 3.1. We say that a function $f \in \mathbb{F}$ is *$\mathcal{D}$-decomposable*, where $\mathcal{D} \in \mathbb{D}$, if and only if there are coefficients $\alpha_A \geq 0$, $A \in \mathcal{D}$, such that

$$\sum_{A \in \mathcal{D}} \alpha_A \mathbb{1}_A = f.$$

Theorem 3.1 (Proposition 3 in [5])**.** *A decomposition integral* $\mathsf{dec}_{\mathcal{H}}$ *with respect to a decomposition system $\mathcal{H} \in \mathbb{H}$ extends the Lebesgue integral if and only if for every function $f \in \mathbb{F}$ there exists a collection $\mathcal{D} \in \mathbb{D}$ such that f is $\mathcal{D}$-decomposable.*

This result is not very practical for checking whether $\mathsf{dec}_{\mathcal{H}}$ extends the Lebesgue integral or not, that is, whether $\mathcal{H} \in \mathbb{H}_+$ or not.

In [5], we obtained the following necessary condition.

Theorem 3.2. *Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system. If $\mathcal{H} \in \mathbb{H}_+$ then for every permutation $\sigma : X \to X$ there exists a collection $\mathcal{D}_\sigma$ such that $\{\sigma(1)\} \in \mathcal{D}_\sigma$ and for every $i \in \{2,3,\dots,n\}$ we have $\{\sigma(i)\} \in \big\{A \setminus \{\sigma(1),\dots,\sigma(i-1)\} : A \in \mathcal{D}_\sigma\big\}$, i.e., for every $i \in \{2,3,\dots,n\}$ there exists $B \subseteq \{\sigma(1),\sigma(2),\dots,\sigma(i-1)\}$ such that $\sigma(i) \cup B \in \mathcal{D}_\sigma$.*

As a corollary (Corollary 1 in [12]) of this theorem we have that

Theorem 3.3. *Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system. If $\mathcal{H} \in \mathbb{H}_+$ then for every $i \in \{1,2,\dots,n\}$ there exists a collection $\mathcal{D}_i$ such that $\{i\} \in \mathcal{D}_i$.*

If we turn our attention to singleton decomposition systems, i.e., $\mathcal{H} = \{\mathcal{D}\}$ for some collection $\mathcal{D} \in \mathbb{D}$, then we obtain that if $\mathcal{H} \in \mathbb{H}_+$ then $\big\{\{1\},\{2\},\dots,\{n\}\big\} = \big\{\{x\} : x \in X\big\} \subseteq \mathcal{D}$. In the case of singleton decomposition systems this is not only a necessary but also a sufficient condition. The following theorem is stated in the form of collection integrals.

Theorem 3.4 (Proposition 2 in [12])**.** *A collection integral* $\mathsf{col}_{\mathcal{D}}$*, where $\mathcal{D} \in \mathbb{D}$ is a collection, extends the Lebesgue integral if and only if $\big\{\{x\} : x \in X\big\} \subseteq \mathcal{D}$.*

It is not only important to be able to check whether a decomposition integral extends the Lebesgue integral, but also to design methods of generating decomposition systems for which the corresponding decomposition integral extends the Lebesgue integral. In the paper [12], we have provided some partition-based construction methods.

Note that we can introduce a partial order $\sqsubseteq$ on the class of all decomposition systems $\mathbb{H}$ given by $\mathcal{H}_1 \sqsubseteq \mathcal{H}_2$ if and only if for every collection $\mathcal{D}_1 \in \mathcal{H}_1$ there exists a collection $\mathcal{D}_2 \in \mathcal{H}_2$ such that $\mathcal{D}_1 \subseteq \mathcal{D}_2$. Now, the property of a decomposition system to belong to $\mathbb{H}_+$ propagates to greater elements with respect to the partial order $\sqsubseteq$:

Theorem 3.5 (Proposition 6 in [12])**.** *Let $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{H}$ be two decomposition systems such that $\mathcal{H}_1 \in \mathbb{H}_+$ and $\mathcal{H}_1 \sqsubseteq \mathcal{H}_2$. Then also $\mathcal{H}_2 \in \mathbb{H}_+$.*

Example 3.1. It is easy to notice that $\mathcal{H}_{\mathsf{Ch}} \sqsubseteq \mathcal{H}_{\mathsf{PC}}$. Also, we know that the Choquet integral extends the Lebesgue integral which implies that $\mathcal{H}_{\mathsf{Ch}} \in \mathbb{H}_+$. Because $\mathcal{H}_{\mathsf{PC}}$ is a greater element than $\mathcal{H}_{\mathsf{Ch}}$, based on the previous theorem we have that $\mathcal{H}_{\mathsf{PC}} \in \mathbb{H}_+$, i.e., the PC integral also extends the Lebesgue integral.

Theorem 3.6 (Theorem 2 in [12])**.** *Let $\{X_1, X_2, \ldots, X_k\}$ be a partition of X and let $\mathcal{H}_{[i]}$ be a decomposition system on X_i for $i = 1, 2, \ldots, k$. Construct a decomposition system $\mathcal{H} \in \mathbb{H}$ by*

$$\mathcal{H} = \left\{ \bigcup_{i=1}^{k} \mathcal{D}_{[i]} : \mathcal{D}_{[i]} \in \mathcal{H}_{[i]} \text{ for } i = 1, 2, \ldots, k \right\}.$$

Then,

$$\mathsf{dec}_{\mathcal{H}}(f, \mu) = \sum_{i=1}^{k} \mathsf{dec}_{\mathcal{H}_{[i]}}(f_{[i]}, \mu_{[i]})$$

for any function $f \in \mathbb{F}$ and any monotone measure $\mu \in \mathbb{M}$, where $f_{[i]} = f \restriction_{X_i}$ and $\mu_{[i]} = \mu \restriction_{2^{X_i}}$ for $i = 1, 2, \ldots, k$.

As a corollary of the previous theorem we obtain a construction method for new decomposition systems belonging to $\mathbb{H}_+$:

Theorem 3.7 (Corollary 3 in [12])**.** *Let $\mathcal{H}_{[i]} \in \mathbb{H}_+$ (for $i = 1, 2, \ldots, k$) and $\mathcal{H} \in \mathbb{H}$ be decomposition systems satisfying the constraints of the previous theorem. Then $\mathcal{H} \in \mathbb{H}_+$.*

This construction method builds a 'larger' decomposition system from smaller ones. In a similar fashion, we can also decompose a larger decomposition system into a smaller one while maintaining the property of belonging to $\mathbb{H}_+$:

Theorem 3.8 (Theorem 3 in [12])**.** *Let $\mathcal{H} \in \mathbb{H}_+$, let $\mathcal{D} \in \mathcal{H}$ be any collection in $\mathcal{H}$, and let $A \in \mathcal{D}$ be any set from the selected collection $\mathcal{D}$. Let $\{A_1, A_2, \ldots, A_k\}$ be a partition of the set A. Construct a new collection $\mathcal{D}_A$ from the collection $\mathcal{D}$ by removing the set A and adding the sets $A_1, A_2, \ldots, A_k$, i.e.,*

$$\mathcal{D}_A = \{A_1, A_2, \ldots, A_k\} \cup \Big(\mathcal{D} \setminus \{A\}\Big).$$

Now construct a new decomposition system $\mathcal{H}_A$ by removing the original collection $\mathcal{D}$ by the new collection $\mathcal{D}_A$, i.e., set

$$\mathcal{H}_A = \{\mathcal{D}_A\} \cup \Big(\mathcal{H} \setminus \{\mathcal{D}\}\Big).$$

Then $\mathcal{H}_A \in \mathbb{H}_+$.

3.2 Decomposition integrals for interval-valued functions

In [15, 20, 22, 23], a possibility of extending decomposition integrals for interval-valued functions was analysed and two extensions were proposed. The first extension is based on the same principle as is the Aumann integral for set-valued functions that extends the Riemann integral, i.e., by creating an envelope of all possible values for all possible integrable real functions constructable from the set-valued function. The second principle is based on a direct modification of the definition of a decomposition integral using interval operations.

An *interval-valued function* is any mapping $v: X \to L([0,\infty[)$, where $L([0,\infty[)$ is the class of all closed intervals that are a subset of the non-negative real line, i.e.,

$$L([0,\infty[) = \{[a,b]: 0 \le a \le b\}.$$

The class of all interval-valued functions will be denoted by $\mathbb{V}$. Note that every interval-valued function v is represented by two functions $a_v, b_v \in \mathbb{F}$ such that $v(x) = [a_v(x), b_v(x)]$ holds for all $x \in X$. The function a_v will be referred to as the *left-endpoint* and the function b_v will be referred to as the *right-endpoint* of the interval-valued function v.

We say that a function $f \in \mathbb{F}$ is contained in an interval-valued function $v \in \mathbb{V}$, writing $f \in v$, if and only if $f(x) \in v(x)$ for all $x \in X$ (note that then f is called a selector of v). Now, we can proceed to the extension of decomposition integrals for interval-valued functions based on the Aumann integral.

Definition 3.2. Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system. An *Aumann-like decomposition integral* is an operator $\overline{\mathsf{dec}}_{\mathcal{H}}: \mathbb{V} \times \mathbb{M} \to 2^{[0,\infty[}$ given by

$$\overline{\mathsf{dec}}_{\mathcal{H}}(v,\mu) = \{\mathsf{dec}_{\mathcal{H}}(f,\mu): f \in v\}$$

for all interval-valued functions $v \in \mathbb{V}$ and all monotone measures $\mu \in \mathbb{M}$.

Theorem 3.9 (Theorem 11 in [15])**.** *Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system. Then,*

$$\overline{\mathsf{dec}}_{\mathcal{H}}(v,\mu) = \left[\mathsf{dec}_{\mathcal{H}}(a_v,\mu), \mathsf{dec}_{\mathcal{H}}(b_v,\mu)\right]$$

for all interval-valued functions $v \in \mathbb{V}$ and all monotone measures $\mu \in \mathbb{M}$.

The second approach was to directly modify the definition of decomposition integrals for interval-valued functions using interval operations. This lead to the following definition:

Definition 3.3. Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system. An *interval-valued decomposition integral* is an operator $\widetilde{\mathsf{dec}}_{\mathcal{H}}: \mathbb{V} \times \mathbb{M} \to 2^{[0,\infty[}$ such that

$$\widetilde{\mathsf{dec}}_{\mathcal{H}}(v,\mu) = \bigvee_{\mathcal{D}\in\mathcal{H}} \bigvee \left\{ \sum_{A\in\mathcal{D}} [\alpha_A, \beta_A]\mu(A): \sum_{A\in\mathcal{D}} [\alpha_A,\beta_A]\mathbb{1}_A \le v, [\alpha_A,\beta_A] \in L([0,\infty[)\ \forall A \in \mathcal{D} \right\}$$

for all interval-valued functions $v \in \mathbb{V}$ and all monotone measures $\mu \in \mathbb{M}$.

Interestingly, with this alternative definition of decomposition integrals for interval-valued functions, it can be proved that this operator is exactly the same as is the Aumann-like decomposition integral, that is, the following theorem holds.

Theorem 3.10 (Theorem 12 in [15])**.** *Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system. Then,*

$$\widetilde{\mathsf{dec}}_{\mathcal{H}}(v,\mu) = \left[\mathsf{dec}_{\mathcal{H}}(a_v,\mu), \mathsf{dec}_{\mathcal{H}}(b_v,\mu)\right]$$

for all interval-valued functions $v \in \mathbb{V}$ and all monotone measures $\mu \in \mathbb{M}$.

We conclude this section with an example for interval-valued decomposition integrals.

Example 3.2 (Example 5 in [15]). Imagine a company which employs three people—two workers and one manager. The considered space is $X = \{1,2,3\}$, where number 1 represents the first worker, 2 represents the second worker, and 3 represents the manager. The company building has a working space where at most two people can work at once. Also, if the manager is at work, then he does not leave for the day. This leads to the following decomposition system

$$\mathcal{H} = \{\{\{1,3\},\{2,3\},\{3\}\},\{\{1\},\{2\},\{1,2\}\}\}.$$

The monotone measure μ encodes the productivity per hour of different groups of people. If workers work individually, then $\mu(\{1\}) = 5$ and $\mu(\{2\}) = 6$. If they work together, then $\mu(\{1,2\}) = 12$. On the other hand, the productivity of the manager is zero, i.e., $\mu(\{3\}) = 0$, and a combination of the manager and a worker results in the increased productivity, e.g., $\mu(\{1,3\}) = \mu(\{2,3\}) = 7$. Working hours of the first worker are between 4 and 6 hours, the second worker's hours are between 5 and 7, and the manager works exactly 5 hours, i.e., $v(1) = [4,6]$, $v(2) = [5,7]$ and $v(3) = [5,5]$. Note that, for this example, $a_v(1) = 4$, $a_v(2) = 5$, and $a_v(3) = 5$. Similarly, $b_v(1) = 6$, $b_v(2) = 7$, and $b_v(3) = 5$. Then,

$$\mathsf{dec}_{\mathcal{H}}(a_v,\mu) = 50 \quad \text{and} \quad \mathsf{dec}_{\mathcal{H}}(b_v,\mu) = 72,$$

and thus $\overline{\mathsf{dec}}_{\mathcal{H}}(v,\mu) = \widetilde{\mathsf{dec}}_{\mathcal{H}}(v,\mu) = [50,72]$.

3.3 Collection integrals

3.3.1 Collection integral and the Choquet integral

In [27, 29] we were interested in finding the collection integrals $\mathsf{col}_{\mathcal{D}}$ that can be represented by the Choquet integral. In other words, we wanted to characterize a sub-class $\mathbb{D}_{\mathsf{Ch}} \subseteq \mathbb{D}$ of collections that consists of collections $\mathcal{D}$ for which there exists a monotone measure $\nu \in \mathbb{M}$ such that $\mathsf{col}_{\mathcal{D}}(\cdot, \mu) = \mathsf{Ch}(\cdot, \nu)$ for all monotone measures $\mu \in \mathbb{M}$. Note that the monotone measure ν is dependent on the monotone measure μ.

A different approach to describe the class $\mathbb{D}_{\mathsf{Ch}}$ is to say that $\mathbb{D}_{\mathsf{Ch}}$ consists of all collections $\mathcal{D} \in \mathbb{D}$ for which the collection integral $\mathsf{col}_{\mathcal{D}}$ is a comonotone-additive operator (in the first argument), i.e., if $f, g \in \mathbb{F}$ are two comonotone functions, then

$$\mathsf{col}_{\mathcal{D}}(f + g, \mu) = \mathsf{col}_{\mathcal{D}}(f, \mu) + \mathsf{col}_{\mathcal{D}}(g, \mu)$$

for all monotone measures $\mu \in \mathbb{M}$, where the choice of the comonotone functions f, g was arbitrary.

Example 3.3. Let us consider a space $X = \{1, 2, 3\}$. It is easy to notice that, e.g.,

$$\mathcal{D}_{\mathsf{con}} = 2^X \setminus \{\varnothing\},$$

i.e., $\mathsf{col}_{\mathcal{D}_{\mathsf{con}}}$ is the concave integral, does not belong to $\mathbb{D}_{\mathsf{Ch}}$. If we consider, e.g., functions $f = (1, 1, 5)$ and $g = (4, 6, 8)$ and a monotone measure $\mu \in \mathbb{M}$ given by $\mu(\{1\}) = \mu(\{2\}) = 1$, $\mu(\{3\}) = 2$, $\mu(\{1, 2\}) = 4$, $\mu(\{1, 3\}) = 5$, $\mu(\{2, 3\}) = \mu(X) = 6$, then we obtain that

$$\begin{aligned}
\mathsf{col}_{\mathcal{D}_{\mathsf{con}}}(f, \mu) &= 3\mu(\{3\}) + \mu(\{1, 3\}) + \mu(\{2, 3\}) = 17, \\
\mathsf{col}_{\mathcal{D}_{\mathsf{con}}}(g, \mu) &= \mu(\{1, 2\}) + 3\mu(\{1, 3\}) + 5\mu(\{2, 3\}) = 49, \\
\mathsf{col}_{\mathcal{D}_{\mathsf{con}}}(f + g, \mu) &= \mu(\{3\}) + 5\mu(\{1, 3\}) + 7\mu(\{2, 3\}) = 69,
\end{aligned}$$

from which it follows that $\mathsf{col}_{\mathcal{D}_{\mathsf{con}}}(f, \mu) + \mathsf{col}_{\mathcal{D}_{\mathsf{con}}}(g, \mu) \neq \mathsf{col}_{\mathcal{D}_{\mathsf{con}}}(f + g, \mu)$ even though the selected functions f, g are comonotone. This implies that the concave integral is not representable by Choquet integral, i.e., $\mathcal{D}_{\mathsf{con}} \notin \mathbb{D}_{\mathsf{Ch}}$.

Example 3.4. On the other hand, consider $\mathcal{D}_{\mathsf{Leb}} = \big\{\{x\} : x \in X\big\}$, i.e., $\mathsf{col}_{\mathcal{D}_{\mathsf{Leb}}}$ is the Lebesgue integral (extended for monotone measures). It is easy to notice that

$$\mathsf{col}_{\mathcal{D}_{\mathsf{Leb}}}(f, \mu) = \sum_{x \in X} f(x)\mu(\{x\})$$

which trivially implies that this operator is additive and also thus comonotone additive. This shows that $\mathcal{D}_{\mathsf{Leb}} \in \mathbb{D}_{\mathsf{Ch}}$, i.e., the class $\mathbb{D}_{\mathsf{Ch}}$ is non-empty. It follows that $\mathsf{col}_{\mathcal{D}_{\mathsf{Leb}}}(\cdot, \mu) = \mathsf{Ch}(\cdot, \nu)$, where the monotone measure $\nu \in \mathbb{M}$ is given by

$$\nu(A) = \sum_{x \in A} \mu(\{x\})$$

for any monotone measure $\mu \in \mathbb{M}$.

A following characterization of the class $\mathbb{D}_{\mathsf{Ch}}$ was obtained.

Theorem 3.11 (Theorem 2 and 3 in [27])**.** *Let $\mathcal{D} \in \mathbb{D}$ be a collection. Then $\mathcal{D} \in \mathbb{D}_{\mathsf{Ch}}$ if and only if for all $A, B \in \mathcal{D}$ one has $A \cap B \in \{A, B, \varnothing\}$.*

3.3.2 Integral inequalities for collection integrals

Integral inequalities can be found in many applications and theoretical results. Perhaps the best known integral inequalities are Chebyshev's, Jensen's, Cauchy's and Hölder's integral inequalities, where the last one is a generalization of Cauchy's integral inequality.

In the theory of non-linear integrals, different modifications of these integral inequalities are considered to reflect the authors' needs. For the purposes of this section, we will consider the following definition.

Definition 3.4. We say that a collection integral $\mathsf{col}_{\mathcal{D}}$ obeys Chebyshev's integral inequality if and only if

$$\mathsf{col}_{\mathcal{D}}(f,\mu)\mathsf{col}_{\mathcal{D}}(g,\mu) \le \mathsf{col}_{\mathcal{D}}(fg,\mu)\mathsf{col}_{\mathcal{D}}(\mathbb{1}_X,\mu)$$

holds for all comonotone functions $f,g \in \mathbb{F}$ and monotone measures $\mu \in \mathbb{M}$; obeys Jensen's integral inequality if and only if

$$\varphi\left(\frac{\mathsf{col}_{\mathcal{D}}(f,\mu)}{\mathsf{col}_{\mathcal{D}}(\mathbb{1}_X,\mu)}\right) \le \frac{\mathsf{col}_{\mathcal{D}}(\varphi\circ f,\mu)}{\mathsf{col}_{\mathcal{D}}(\mathbb{1}_X,\mu)}$$

holds for all functions $f \in \mathbb{F}$, all monotone measures $\mu \in \mathbb{M}$ and all convex and non-decreasing functions $\varphi\colon [0,\infty[\to[0,\infty[$; obeys Hölder's integral inequality if and only if

$$\mathsf{col}_{\mathcal{D}}(fg,\mu) \le \sqrt[p]{\mathsf{col}_{\mathcal{D}}(f^p,\mu)}\,\sqrt[q]{\mathsf{col}_{\mathcal{D}}(g^q,\mu)}$$

holds for all functions $f,g \in \mathbb{F}$, all monotone measures $\mu \in \mathbb{M}$ and all real numbers $p,q \ge 1$ such that $1/p+1/q = 1$. As a special case of Hölder's integral inequality, setting $p = q = 2$, we obtain the Cauchy's integral inequality.

Note that the Choquet integral obeys all of the aforementioned integral inequalities, see [6, 14]. This implies that all collection integrals representable by the Choquet integral also obey these integral inequalities. It can also be shown that only such collection integrals do obey them.

Theorem 3.12 (Theorem 4 in [27])**.** *A collection integral* $\mathsf{col}_{\mathcal{D}}$ *obeys Chebyshev's (Jensen's, Cauchy's, or Hölder's) integral inequality if and only if* $\mathcal{D} \in \mathbb{D}_{\mathsf{Ch}}$.

3.3.3 Applications of collection integrals

In the paper [3], we examined a possible use of collection integrals in the construction of coherent lower previsions. Coherent lower previsions are generalizations of expected values and are used in the theory of imprecise probabilities. Analogously to coherent lower previsions, similar generalizations are coherent upper previsions.

Let us remind the reader that X is a finite space. For the purposes of this section, we will consider σ-algebra 2^X. Then $\mathbb{F}_{\mathbb{R}}$ (the class of all functions $X \to \mathbb{R}$) can be represented as a set of all random variables on the measurable space $(X, 2^X)$.

Definition 3.5. A *coherent lower prevision* is an operator $\mathrm{clp}\colon \mathbb{F}_{\mathbb{R}} \to \mathbb{R}$ for which the conditions (i) $\mathrm{clp}(f) \ge \inf f(X)$; (ii) $\mathrm{clp}(\lambda f) = \lambda \mathrm{clp}(f)$; and (iii) $\mathrm{clp}(f+g) \ge \mathrm{clp}(f) + \mathrm{clp}(g)$ hold for all functions $f,g \in \mathbb{F}_{\mathbb{R}}$ and all non-negative numbers $\lambda \ge 0$.

In other words, coherent lower previsions are positively-homogeneous super-additive operators bounded below by the minimum value of the argument function. Coherent upper previsions, cup, are defined analogously to coherent lower previsions, replacing the property (i) with a property being bounded above by a maximum value of the argument function and the property (iii) by sub-additivity. Note that there is one-to-one correspondence between coherent lower and upper previsions given by the conjugacy property: The functional clp is a coherent lower prevision if and only if cup is a coherent upper prevision, where $\mathrm{cup}(f) = -\mathrm{clp}(-f)$ for all functions $f \in \mathbb{F}_{\mathbb{R}}$.

The motivation to use collection integrals for constructing coherent lower previsions comes from the super-additivity of collection integrals which is one of the axioms of coherent lower previsions.

Theorem 3.13 (Proposition 1 in [3])**.** *Let* $\mathcal{D} \in \mathbb{D}$ *be any collection. Then* $\mathsf{col}_{\mathcal{D}}$ *is a super-additive operator.*

The first problem that occurs when considering collection integrals for constructing coherent lower previsions is that they are defined only for non-negative functions $\mathbb{F}$ and coherent lower previsions are defined for all real-valued functions $\mathbb{F}_{\mathbb{R}}$. Thus, an extension of collection integrals must be introduced, while maintaining the super-additivity. For such extension, only shift-invariant collection integrals can be considered. We say that a collection integral $\mathsf{col}_{\mathcal{D}}$ is shift-invariant if and only if

$$\mathsf{col}_{\mathcal{D}}(f+\alpha\mathbb{1}_X,\mu) = \mathsf{col}_{\mathcal{D}}(f,\mu) + \alpha\,\mathsf{col}_{\mathcal{D}}(\mathbb{1}_X,\mu)$$

for all functions $f \in \mathbb{F}$ and all non-negative real numbers $\alpha \ge 0$ (note that $\mu \in \mathbb{M}$ is a fixed monotone measure). Now, we can introduce the super-additive extension of collection integrals.

Definition 3.6. Let $\mathcal{D} \in \mathbb{D}$ be a collection and $\mu \in \mathbb{M}$ be a monotone measure such that $\mathsf{col}_{\mathcal{D}}(\cdot, \mu)$ is a shift-invariant collection integral. Then the mapping $\underline{\mathsf{col}}_{\mathcal{D}}(\cdot, \mu) \colon \mathbb{F}_{\mathbb{R}} \to \mathbb{R}$ defined by

$$\underline{\mathsf{col}}_{\mathcal{D}}(f, \mu) = \mathsf{col}_{\mathcal{D}}\Big(f - \big(\inf f(X)\big)\mathbb{1}_X, \mu\Big) + \big(\inf f(X)\big)\mathsf{col}_{\mathcal{D}}(\mathbb{1}_X, \mu)$$

for all functions $f \in \mathbb{F}_{\mathbb{R}}$, is called a *super-additive integral.*

Now, we summarize some of the properties of the super-additive integral.

Theorem 3.14 (Theorem 1 in [3])**.** *Let $\underline{\mathsf{col}}_{\mathcal{D}}(\cdot, \mu)$ be a super-additive integral. Then:*

- $\underline{\mathsf{col}}_{\mathcal{D}}$ *extends* $\mathsf{col}_{\mathcal{D}}$*;*
- $\underline{\mathsf{col}}_{\mathcal{D}}$ *is positively homogeneous;*
- $\underline{\mathsf{col}}_{\mathcal{D}}$ *is shift-invariant;*
- $\underline{\mathsf{col}}_{\mathcal{D}}$ *is bounded below by a normed infimum, i.e.,* $\underline{\mathsf{col}}_{\mathcal{D}}(f, \mu) \geq \big(\inf f(X)\big)\underline{\mathsf{col}}_{\mathcal{D}}(\mathbb{1}_X, \mu)$*;*
- $\underline{\mathsf{col}}_{\mathcal{D}}$ *is super-additive.*

Now, the super-additive integral has almost all the properties of the coherent lower previsions up to property (i). The lack of this property can be resolved by introducing a normalization factor to the definition.

Theorem 3.15 (Theorem 2 in [3])**.** *Let $\mathcal{D} \in \mathbb{D}$ be a collection and let $\mu \in \mathbb{M}$ be a monotone measure such that $\underline{\mathsf{col}}_{\mathcal{D}}(\cdot, \mu)$ is a super-additive integral. Then the mapping*

$$\mathrm{clp}^{\mu}_{\mathcal{D}}(f) = \frac{\underline{\mathsf{col}}_{\mathcal{D}}(f, \mu)}{\underline{\mathsf{col}}_{\mathcal{D}}(\mathbb{1}_X, \mu)},$$

given for all functions $f \in \mathbb{F}_{\mathbb{R}}$, is a coherent lower prevision.

Example 3.5 (Example 4 in [3]). Let us consider a collection $\mathcal{D} = \{A_i\}_{i=1}^k$ of disjoint system of sets. Then, the corresponding coherent lower prevision is given by

$$\mathrm{clp}^{\mu}_{\mathcal{D}}(f) = \frac{\sum\limits_{i=1}^{k} \mu(A_i) \min f(A_i)}{\sum\limits_{i=1}^{k} \mu(A_i)},$$

i.e., $\mathrm{clp}^{\mu}_{\mathcal{D}}$ is a weighted average of the minimal values obtained on sets A_i; the corresponding coherent upper prevision (obtained by the conjugacy property) is then

$$\mathrm{cup}^{\mu}_{\mathcal{D}}(f) = \frac{\sum\limits_{i=1}^{k} \mu(A_i) \max f(A_i)}{\sum\limits_{i=1}^{k} \mu(A_i)},$$

i.e., a weighted average of the maximal values obtained on the sets A_i.

A more general approach, considering aggregation functions instead of collection integrals, was examined in [4].

3.4 Greedy decomposition integrals

The main disadvantage of decomposition integrals is their computational complexity. For example, as noted in [18], the computation of the concave integral is a NP-hard problem, moreover, the solution to the concave integration is not verifiable in a polynomial computational time. To overcome this disadvantage, we proposed greedy decomposition integrals [21].

We start with the definition of a greedy collection integral which will be used as a basic building block of greedy decomposition integrals in the same way as collection integrals are used to build decomposition integrals. The greedy collection integral is defined recursively as follows.

Definition 3.7. Let $\mathcal{D} \in \mathbb{D}$ be a collection. A *greedy collection integral* (with respect to a collection $\mathcal{D}$) is an operator $\mathsf{gcol}_{\mathcal{D}}: \mathbb{F} \times \mathbb{M} \to [0, \infty[$ given by:

- if $\mathcal{D} = \{A\}$ is a singleton collection, then $\mathsf{gcol}_{\mathcal{D}}(f, \mu) = \mu(A) \cdot \min f(A)$; or
- if the collection $\mathcal{D}$ consists of two or more sets, i.e., $|\mathcal{D}| \geq 2$, then we find a real number

$$\alpha = \bigvee_{A \in \mathcal{D}} \mu(A) \cdot \min f(A),$$

we construct a sub-collection $\mathcal{S} \subseteq \mathcal{D}$ by

$$\mathcal{S} = \left\{A \in \mathcal{D}: \mu(A) \cdot \min f(A) = \alpha\right\},$$

and we set

$$\mathsf{gcol}_{\mathcal{D}}(f, \mu) = \alpha + \bigvee_{B \in \mathcal{S}} \mathsf{gcol}_{\mathcal{D} \setminus \{B\}}\Big(f - \mathbb{1}_B \cdot \min f(B), \mu\Big).$$

Definition 3.8. Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system. A *greedy decomposition integral* (with respect to a decomposition system $\mathcal{H}$) is an operator $\mathsf{gdec}_{\mathcal{H}}: \mathbb{F} \times \mathbb{M} \to [0, \infty[$ given by

$$\mathsf{gdec}_{\mathcal{H}}(f, \mu) = \bigvee_{\mathcal{D} \in \mathcal{H}} \mathsf{gcol}_{\mathcal{D}}(f, \mu)$$

for all functions $f \in \mathbb{F}$ and all monotone measures $\mu \in \mathbb{M}$.

A greedy collection (decomposition) integral is well-defined for all collections, functions, and all monotone measures (Proposition 3.2 in [21]). Unfortunately, due to its greedy nature, the operator gcol lacks the property of being monotone in both arguments (Proposition 3.8 in [21]) as seen in following examples.

Example 3.6 (Example 3.6 in [21]). Let $\mathcal{D} = \{\{1,2,3\}, \{2,3\}, \{3\}\}$ be a collection and let $\mu \in \mathbb{M}$ be a monotone measure such that $\mu(\{1,2,3\}) = 2$, $\mu(\{2,3\}) = 1$, and $\mu(\{3\}) = 1/4$. Consider two functions $f, g \in \mathbb{F}$, $f = (1, 1.9, 3)$ and $g = (1, 2.2, 3)$. Then $f \leq g$, but

$$\mathsf{gcol}_{\mathcal{D}}(f, \mu) = 3.175 \qquad \text{and} \qquad \mathsf{gcol}_{\mathcal{D}}(g, \mu) = 2.4$$

which shows that $\mathsf{gcol}_{\mathcal{D}}$ is not monotone in the first argument, in general.

Example 3.7 (Example 3.7 in [21]). Now consider collection $\mathcal{D} = \{\{1,2,3\}, \{2,3\}, \{3\}\}$ and let $f \in \mathbb{F}$ be a function such that $f = (1,2,3)$. Consider two monotone measures $\mu, \nu \in \mathbb{M}$ given by $\mu(\{1,2,3\}) = \nu(\{1,2,3\}) = 2$, $\mu(\{2,3\}) = 0.9$, $\nu(\{2,3\}) = 1.1$, and $\mu(\{3\}) = \nu(\{3\}) = 1/4$. Then $\mu \leq \nu$, but

$$\mathsf{gcol}_{\mathcal{D}}(f, \mu) = 3.15 \qquad \text{and} \qquad \mathsf{gcol}_{\mathcal{D}}(f, \nu) = 2.45,$$

which shows that $\mathsf{gcol}_{\mathcal{D}}$ is not monotone in the second argument, in general, either.

Moreover, it can be proved that $\mathsf{gcol}_{\mathcal{D}}$ is monotone if and only if $\mathcal{D}$ is a disjoint system and then $\mathsf{gcol}_{\mathcal{D}}$ coincides with $\mathsf{col}_{\mathcal{D}}$ (Corollaries 3.19 and 3.20 in [21]). All of the previous results apply also for the greedy decomposition integral.

3.5 Computation of specific decomposition integrals

Results presented in this section come from our publications [18, 19].

Concave integral

Note that any collection integral with respect to a collection $\mathcal{D} \in \mathbb{D}$ can be viewed as a linear optimization problem with $|\mathcal{D}|$ unknown variables and $|X| = n$ linear constraints. For example, in the case of the concave integral we have $|\mathcal{D}| = 2^n - 1$. It can be shown, see [18], that the problem of concave integration (in the form of a linear optimization problem) is harder than the problems from the NP computational class.

Choquet and chain integrals

Both the Choquet integral and the chain integral (chain integral is a collection integral with respect to a collection that forms a chain, see, e.g., [19]) can be computed by ordering the input vectors (or, equivalently, by ordering the values of the input function). This can be done in $O(n \log n)$ computational steps and thus the problem of computing the Choquet and chain integrals can be reduced to $O(n \log n)$ algorithms.

Min-max integral

The min-max integral, introduced in [19], is a collection integral with respect to a collection $\mathcal{D} = \{X\}$. The value of such integral for a function $f \in \mathbb{F}$ and a monotone measure $\mu \in \mathbb{M}$ is given by

$$\mu(X) \min_{x \in X} f(x),$$

and the only thing to be computed is the minimum of the values of the function f. This can be done in $O(n)$ steps and thus the computation of the value of the min-max integral will take also $O(n)$ steps.

Shilkret and PAN integrals

Neither for the Shilkret nor the PAN integrals, the computation of their values is easy. In general, we can compute their values by using brute force algorithms only, i.e., algorithms that check all possible combinations. Fortunately, there exist polynomial verifiers for both of these brute force algorithms [19] and thus the computation belongs to at most NP computational class of problems. The brute force algorithm for the Shilkret integral takes $O(2^n n)$ steps and for the PAN integral $O(3^n n^2)$ steps.

Chapter 4

Results on aggregation functions

4.1 Transformations of aggregation functions

4.1.1 k-bounded transformations

In publications [9, 17, 24], we investigated a new transformation of aggregation functions, called k-bounded transformations, in collaboration with Katarína Hriňáková.

The super- and sub-additive transformations allow the decomposition of the input vector $\mathbf{x}$ into infinitely many non-negative vectors $\mathbf{x}_i$ summing up to $\mathbf{x}$. In practice, this condition of unboundedness is impossible and thus a restriction on the number of decomposition points should be placed. This lead to the definition of k-bounded transformations.

Definition 4.1. Let $\mathsf{A} \in \mathbb{A}$ be an aggregation function and let $k \in \mathbb{N}$ be a natural number. An *upper k-bounded transformation* of A is a mapping $\mathsf{A}^{(k)}$, with the same domain and co-domain as A has, given by

$$\mathsf{A}^{(k)}(\mathbf{x}) = \bigvee \left\{ \sum_{i=1}^{k} \mathsf{A}(\mathbf{x}_i) \colon \sum_{i=1}^{k} \mathbf{x}_i = \mathbf{x}, \mathbf{x}_i \geq \mathbf{0} \right\}$$

for all $\mathbf{x} \geq \mathbf{0}$. Analogously, a *lower k-bounded transformation* of A is a mapping $\mathsf{A}_{(k)}$ defined by

$$\mathsf{A}_{(k)}(\mathbf{x}) = \bigwedge \left\{ \sum_{i=1}^{k} \mathsf{A}(\mathbf{x}_i) \colon \sum_{i=1}^{k} \mathbf{x}_i = \mathbf{x}, \mathbf{x}_i \geq \mathbf{0} \right\}.$$

It can be easily proved that the output of both the upper and lower k-bounded transformations is again an aggregation function. In contrast with the super- and sub-additive transformations of aggregation functions, these new transformations do not lead to super-additive nor sub-additive aggregation functions, in general.

Example 4.1 (Example 3.2 in [9]). If we turn our attention to one-dimensional aggregation functions F, G and H introduced in Example 2.1, we obtain

$$\mathsf{F}^{(k)}(x) = \min\{x, k\}, \quad \mathsf{G}^{(k)}(x) = \sqrt{kx} \quad \text{and} \quad \mathsf{H}^{(k)}(x) = x^2$$

for the upper k-bounded transformation and

$$\mathsf{F}_{(k)}(x) = \min\{x, 1\}, \quad \mathsf{G}_{(k)}(x) = \sqrt{x} \quad \text{and} \quad \mathsf{H}_{(k)}(x) = \frac{x^2}{k}$$

for the lower k-bounded transformation.

The following rule of composition of k-bounded transformations holds.

Theorem 4.1 (Proposition 3.5 in [9])**.** *Let $\mathsf{A} \in \mathbb{A}$ be an aggregation function and let $k, l \in \mathbb{N}$ be two natural numbers. Then,*

$$\left(\mathsf{A}^{(k)}\right)^{(l)} = \mathsf{A}^{(kl)} \quad \textit{and} \quad \left(\mathsf{A}_{(k)}\right)_{(l)} = \mathsf{A}_{(kl)}.$$

In the following theorem we summarize some of the properties of k-bounded transformations of aggregation functions.

Theorem 4.2 (Theorems 3.6, 3.7 and Proposition 3.8 in [9]). *Let* $\mathsf{A}, \mathsf{B} \in \mathbb{A}$ *be two aggregation functions,* $k \in \mathbb{N}$ *be a natural number and let* $\alpha \geq 0$ *be a non-negative real number. Then,*

- *if* $\mathsf{A} \leq \mathsf{B}$*, then* $\mathsf{A}^{(k)} \leq \mathsf{B}^{(k)}$ *and* $\mathsf{A}_{(k)} \leq \mathsf{B}_{(k)}$*;*
- *if* A *is super-additive, then* $\mathsf{A}^{(k)} = \mathsf{A}$*;*
- *if* A *is sub-additive, then* $\mathsf{A}_{(k)} = \mathsf{A}$*;*
- $\cdot^{(k)}$ *and* $\cdot_{(k)}$ *are positively homogeneous operators of degree 1, i.e., the following equalities* $(\alpha\mathsf{C})^{(k)} = \alpha\mathsf{C}^{(k)}$ *and* $(\alpha\mathsf{C})_{(k)} = \alpha\mathsf{C}_{(k)}$ *hold for all* $\mathsf{C} \in \mathbb{A}$*;*
- $\mathsf{A}^{(1)} = \mathsf{A}$ *and* $\mathsf{A}^{(k+1)} \geq \mathsf{A}^{(k)}$*;*
- $\mathsf{A}_{(1)} = \mathsf{A}$ *and* $\mathsf{A}_{(k+1)} \leq \mathsf{A}_{(k)}$*; and*
- *if* A *is continuous, then so are* $\mathsf{A}^{(k)}$ *and* $\mathsf{A}_{(k)}$*.*

4.1.2 Continuity lifts of super- and sub-additive transformations

In our papers [9, 25, 26], we examined the types of continuity of aggregation functions that are preserved by their super- and sub-additive transformations. First of all, let us start by summarizing different types of continuities of functions: A function $f\colon [0,\infty[^n \to [0,\infty[$ is called

- *continuous*, if for every $\mathbf{x}, \mathbf{y} \in [0,\infty[^n$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\|\mathbf{x} - \mathbf{y}\| < \delta \quad \text{implies} \quad |f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon;$$
- *uniformly continuous*, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\|\mathbf{x} - \mathbf{y}\| < \delta \quad \text{implies} \quad |f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$$
holds for all $\mathbf{x}, \mathbf{y} \in [0,\infty[^n$;
- *α-Hölder continuous*, if there exists a non-negative constant $c > 0$ such that
$$|f(x) - f(y)| \leq c\,\|\mathbf{x} - \mathbf{y}\|^{\alpha}$$
for every $\mathbf{x}, \mathbf{y} \in [0,\infty[^n$;
- *Lipschitz continuous*, if a function f is 1-Hölder continuous.

The following results were obtained.

Theorem 4.3 (Theorem 4.3 in [9]). *If* $\mathsf{A} \in \mathbb{A}$ *is a continuous aggregation function then so is* A^* *(if* A *does not escape) and* A_**.*

Theorem 4.4 (Theorems 1 and 2 in [25]). *If* $\mathsf{A} \in \mathbb{A}$ *is a Lipschitz continuous aggregation function then so is* A^* *(if* A *does not escape) and* A_**.*

Note that if A is a Lipschitz continuous aggregation function with a Lipschitz' constant c, then both A^* and $\mathbf{A}_*$ are Lipschitz continuous aggregation functions with a Lipschitz' constant equal to $c\sqrt{n}$, where n is the dimension of the aggregation function A.

Theorem 4.5. *If* $\mathsf{A} \in \mathbb{A}$ *is an* α*-Hölder continuous aggregation function, where* $\alpha \in]0,1[$*, then* A^* *(if* A^* *does not escape) nor* A_* *is not* β*-Hölder continuous for any* $\beta \in]0,1[$*, in general.*

Example 4.2. Let $\alpha \in]0,1[$ be a real number and consider an aggregation function

$$\mathsf{A}(\mathbf{x}) = \Big(1 + \|\mathbf{x}\|\Big)^{\alpha} - 1.$$

It can be shown that this aggregation function is α-Hölder continuous (see discussion in [25]), but its super-additive transformation $\mathsf{A}^*(\mathbf{x}) = \alpha(x_1 + x_2 + \cdots + x_n)$, where $n \in \mathbb{N}$ is the dimension of the aggregation function A, is not β-Hölder continuous for any $\beta \in]0,1[$. An analogous counter-example can be constructed for the case of sub-additive transformation.

Now, we turn our attention to uniform continuity. We were able to prove the uniform continuity preservation in the case of the sub-additive transformation, which is a stronger variant than the case of the super-additive transformation.

Theorem 4.6 (Theorem 3 in [25])**.** *Let $\mathsf{A} \in \mathbb{A}$ be an aggregation function that is continuous at the origin. Then A_* is a uniformly continuous aggregation function.*

Lastly, we were able to prove the uniform continuity preservation in the case of the super-additive transformation of aggregation functions only for one-dimensional aggregation functions.

Theorem 4.7 (Theorem 4 in [25])**.** *Let $\mathsf{A} \in \mathbb{A}$ be a one-dimensional uniformly continuous aggregation function such that A^* does not escape. Then, A^* is also uniformly continuous.*

Whether the previous theorem holds for any dimensional aggregation functions remains an interesting challenge. We were not able to prove nor contradict such theorem.

4.1.3 Revenue transformation

The revenue transformation of aggregation functions is based on the interpretation of an aggregation function being a mapping transforming resources to incomes. We can ask the following question: "What is the maximal increase in incomes if we increase our resources by some value $\mathbf{x}$?". Since we do not consider to have a particular number of resources at our hand, the answer to this question will be

$$\max_{\mathbf{y}\geq\mathbf{0}} \Big(\mathsf{A}(\mathbf{y}+\mathbf{x}) - \mathsf{A}(\mathbf{y})\Big).$$

There are two difficulties with this definition. First, the maximum might not exist and thus it must be replaced by supremum. The second one is that the supremum might be equal to infinity which we would like to avoid. So in this section we will consider only those aggregation functions $\mathsf{A}\colon [0,\infty[^n \to [0,\infty[$ for which

$$\bigvee_{\mathbf{y}\geq\mathbf{0}} \Big(\mathsf{A}(\mathbf{y}+\mathbf{x}) - \mathsf{A}(\mathbf{y})\Big) < \infty$$

for all $\mathbf{x} \geq \mathbf{0}$. The class of all such aggregation functions will be denoted by $\mathbb{A}_{\mathsf{rev}}$. The revenue transformation and its properties were discussed in our paper [4]. We recommend to an interested reader to consult the discussion and the proofs of the following definition and theorem in the paper [4].

Definition 4.2. Let $\mathsf{A}\colon [0,\infty[^n \to [0,\infty[$ be an aggregation function from $\mathbb{A}_{\mathsf{rev}}$. Its revenue transformation is a mapping $\overline{\mathsf{A}}\colon [0,\infty[^n \to [0,\infty[$ given by

$$\overline{\mathsf{A}}(\mathbf{x}) = \bigvee_{\mathbf{y}\geq\mathbf{0}} \Big(\mathsf{A}(\mathbf{y}+\mathbf{x}) - \mathsf{A}(\mathbf{y})\Big)$$

for all $\mathbf{x} \in [0,\infty[^n$.

In the following theorem, a summary of some properties of the revenue transformation of an aggregation function is given.

Theorem 4.8 (Propositions 1 and 2 in [4]). *Let* $\mathsf{A} \in \mathbb{A}_{\mathsf{rev}}$ *be an aggregation function. Then*

- $\overline{\mathsf{A}}$ *is an aggregation function;*
- $\overline{\mathsf{A}}$ *is sub-additive;*
- $\overline{\mathsf{A}} \geq \mathsf{A}$*;*
- *if* A *is sub-additive, then* $\overline{\mathsf{A}} = \mathsf{A}$*;*
- *if* A *is shift-invariant, then also* $\overline{\mathsf{A}}$ *is and the diagonal of* $\overline{\mathsf{A}}$ *coincides with the diagonal of* A*; and*
- *if* A *is positively homogeneous, then so is* $\overline{\mathsf{A}}$.

Example 4.3. If we consider the one-dimensional aggregation functions from Example 2.1, then one obtains that $\overline{\mathsf{F}} = \mathsf{F}$, $\overline{\mathsf{G}} = \mathsf{G}$ and $\mathsf{H} \notin \mathbb{A}_{\mathsf{rev}}$, i.e., $\overline{\mathsf{H}}$ is not well-defined. If we consider, e.g., a one-dimensional aggregation function $\mathsf{A}(x) = x^p$, where $p > 0$, then $\overline{\mathsf{A}} = \mathsf{A}$ if $0 < p \leq 1$ and, if $p > 1$, then

$$\overline{\mathsf{A}}(x) = \begin{cases} 1 - (1-x)^p, & \text{if } x \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Example 4.4. If we consider a two-dimensional Choquet integral (on the space $X = \{1,2\}$) with respect to a monotone measure $\mu \in \mathbb{M}$, then we obtain

$$\overline{\mathsf{Ch}}(\cdot,\mu) = \mathsf{Ch}(\cdot,\nu),$$

where $\nu \in \mathbb{N}$ is a monotone measure given by

$$\nu(\{x\}) = \mu(\{x\}) + \max\Big\{0, \mu(\{1,2\}) - \mu(\{1\}) - \mu(\{2\})\Big\} \quad \text{and} \quad \nu(\{1,2\}) = \mu(\{1,2\})$$

for $x \in \{1,2\}$.

It is an open problem whether $\overline{\mathsf{Ch}}(\cdot,\mu)$ is again a Choquet integral for any dimension. Also, note that, e.g., the property $\mathsf{A} \leq \mathsf{B}$ implies $\overline{\mathsf{A}} \leq \overline{\mathsf{B}}$ does not hold in general. Consider, e.g., A to be the minimum and B the arithmetic mean, then $\overline{\mathsf{A}}$ is the maximum and $\overline{\mathsf{B}} = \mathsf{B}$. In other words, the revenue transformation is not order-preserving transformation.

4.1.4 Other transformation of aggregation functions

In our paper [9], we also introduced different types of transformations of aggregation functions (alongside with the k-bounded transformations) that are direct modifications of the super- and sub-additive transformations.

Super-additive ray transformation

The first modification is a restriction of the points that can be used to decompose the input vector $\mathbf{x}$. We restrict ourselves only to the points that lie on a line segment joining points $\mathbf{0}$ and $\mathbf{x}$, i.e., points $\lambda\mathbf{x}$, where $\lambda \in [0,1]$.

Definition 4.3. Let $\mathsf{A}\colon [0,\infty[^n \to [0,\infty[$ be an aggregation function. Its *super-additive ray transformation* is a mapping $\mathsf{A}^{-}\colon [0,\infty[^n \to [0,\infty[$ given by

$$\mathsf{A}^{-}(\mathbf{x}) = \bigvee \left\{ \sum_{i=1}^{k} \mathsf{A}(\lambda_i \mathbf{x}) \colon \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1, k \in \mathbb{N} \right\}$$

for every $\mathbf{x} \in [0,\infty[^n$.

Note that it might be the case that A^{-} escapes as in the case of the super-additive transformation. E.g., consider a one-dimensional aggregation function G from Example 2.1, i.e., $\mathsf{G}(x) = \sqrt{x}$, then $\mathsf{G}^{-}(x) = \infty$ for $x > 0$. The term *escapeness* is thus naturally adopted for the super-additive ray transformations.

Theorem 4.9 (Theorems 5.2, 5.3 and 5.5 in [9]). *Let $\mathsf{A} \in \mathbb{A}$ be an aggregation function such that A^- does not escape. Then, A^- is again an aggregation function such that $\mathsf{A} \le \mathsf{A}^- \le \mathsf{A}^*$, and*

$$\mathsf{A}^-\Big((\alpha+\beta)\mathbf{x}\Big) \ge \mathsf{A}^-(\alpha\mathbf{x}) + \mathsf{A}^-(\beta\mathbf{x})$$

for all $\mathbf{x} \in [0,\infty[^n$ and all $\alpha,\beta \ge 0$, i.e., A^- is super-additive on the rays starting at $\mathbf{0}$.

Example 4.5. Note that A^- is not super-additive, in general. Consider the two-dimensional aggregation function K from Example 2.2. Then, it is easy to check that $\mathsf{K}^- = \mathsf{K}$, but K is not super-additive because, e.g.,

$$\mathsf{K}(3,4) = 5 \not\ge 7 = 3 + 4 = \mathsf{K}(3,0) + \mathsf{K}(0,4).$$

There is a relation between the escapeness related to the super-additive transformation and the escapeness related to the super-additive ray transformation.

Theorem 4.10. *Let $\mathsf{A} \in \mathbb{A}$ be an aggregation function. Then, A^- escapes if and only if A^* escapes.*

The following two theorems are corollaries of the previous theorem.

Theorem 4.11. *Let $\mathsf{A} \in \mathbb{A}$ be an aggregation function such that A^- does not escape. Then,*

$$\mathsf{A}^-(\mathbf{x}) \le \mathsf{A}^*(\mathbf{x}) \le \mathsf{A}^-\left(\left(\sum_{i=1}^{n} x_i\right)\mathbf{1}\right)$$

for every $\mathbf{x} \ge \mathbf{0}$.

Theorem 4.12. *Let $\mathsf{A} \in \mathbb{A}$ be a one-dimensional aggregation function such that A^- does not escape. Then, A^* and A^- coincide.*

Linear super-additive transformation

In the case of super-additive transformations, we consider only addition of non-negative vectors $\mathbf{x}_i$ summing to the input vector $\mathbf{x}$. If we loosen this condition and we consider all linear combinations of non-negative vectors summing to the input vector, we obtain the definition of linear super-additive transformations.

Definition 4.4. Let $\mathsf{A}: [0,\infty[^n \to [0,\infty[$ be an aggregation function. Its *linear super-additive transformation* is a mapping $\mathsf{A}^\dagger: [0,\infty[^n \to [0,\infty[$ defined by

$$\mathsf{A}^\dagger(\mathbf{x}) = \bigvee \left\{ \sum_{i=1}^{k} \alpha_i \mathsf{A}(\mathbf{x}_i) : \alpha_i \ge 0, \mathbf{x}_i \ge \mathbf{0}, \sum_{i=1}^{k} \alpha_i \mathbf{x}_i = \mathbf{x}, k \in \mathbb{N} \right\}$$

for every $\mathbf{x} \in [0,\infty[^n$.

Again, the concept of *escapeness* is adopted also for the linear super-additive transformation.

Theorem 4.13 (Theorems 5.7 and 5.8 in [9]). *Let $\mathsf{A} \in \mathbb{A}$ be an aggregation function such that $\mathsf{A}^\dagger$ does not escape. Then $\mathsf{A}^\dagger$ is super-additive aggregation function such that $\mathsf{A}^* \le \mathsf{A}^\dagger$.*

Example 4.6. Note that it may happen that A^* does not escape, but $\mathsf{A}^\dagger$ escapes. Consider, e.g., the aggregation function L from Example 2.2. Then $\mathsf{L}^* = \mathsf{L}$ but, e.g., $\mathsf{L}^\dagger(1,1) = \infty$, because

$$\mathsf{L}^\dagger(1,1) \ge \bigvee \left\{ \sum_{i=1}^{k} \frac{1}{k} : k \in \mathbb{N} \right\} = \sum_{i=1}^{\infty} \frac{1}{k} = \infty,$$

which we obtain by choosing points $\mathbf{x}_i = (1,1)$ and coefficients $\alpha_i = 1/k$ for all $i = 1,2,\ldots,k$, where $k \in \mathbb{N}$.

Interestingly, if the linear super-additive transformation exists, then this transformation is a positively homogeneous aggregation function.

Theorem 4.14. *Let $\mathsf{A} \in \mathbb{A}$ be an aggregation function such that $\mathsf{A}^\dagger$ does not escape. Then, $\mathsf{A}^\dagger$ is positively homogeneous.*

Analogous modifications can be introduced also for the sub-additive transformation of aggregation functions and are a subject of our future research.

4.2 Convolution of aggregation functions

A convolution, as a process of combining two functions into one, turned out to be a useful tool in integral and differential theories, or in image processing. In our paper [16], we introduced four different ways of convolution of aggregation functions and all of them were heavily motivated by the following example.

Example 4.7. Let us consider two aggregation functions $\mathsf{A}, \mathsf{B}\colon [0,\infty[^n \to [0,\infty[$. Now, let $\mathbf{x} = (x_1, \dots, x_n)$ represent a vector of quantities of n different products. Let $\mathsf{A}(\mathbf{x})$ represent the price that the first consumer is willing to pay for products $\mathbf{x}$, and, analogously, let $\mathsf{B}(\mathbf{x})$ represent the price that the second consumer is willing to pay for products $\mathbf{x}$. What is the maximum price that we can obtain by dividing the resources $\mathbf{x}$ between the two consumers?

Similarly, we can consider a minimization problem instead of maximization one. This leads to the following definitions of convolution.

Definition 4.5. Let $\mathsf{A}\colon [0,\infty[^n \to [0,\infty[$ and $\mathsf{B}\colon [0,\infty[^n \to [0,\infty[$ be two aggregation functions. An (upper) *convolution* of these aggregation functions is a mapping $\mathsf{A} \triangledown \mathsf{B}\colon [0,\infty[^n \to [0,\infty[$ such that

$$(A \triangledown B)(\mathbf{x}) = \sup \{\mathsf{A}(\mathbf{y}) + \mathsf{B}(\mathbf{z})\colon \mathbf{y}, \mathbf{z} \in [0,\infty[^n, \mathbf{y} + \mathbf{z} = \mathbf{x}\}.$$

A *lower convolution* is a mapping $\mathsf{A} \vartriangle \mathsf{B}\colon [0,\infty[^n \to [0,\infty[$ such that

$$(A \vartriangle B)(\mathbf{x}) = \inf \{\mathsf{A}(\mathbf{y}) + \mathsf{B}(\mathbf{z})\colon \mathbf{y}, \mathbf{z} \in [0,\infty[^n, \mathbf{y} + \mathbf{z} = \mathbf{x}\}.$$

The properties of convolutions defined in this manner are summarized in the following theorem.

Theorem 4.15 (Propositions 3.6, 3.8 and 3.9 in [16]). *Let* $\mathsf{A}, \mathsf{B} \in \mathbb{A}$ *be two aggregation functions. Then*

- $\mathsf{A} \triangledown \mathsf{B}$ *and* $\mathsf{A} \vartriangle \mathsf{B}$ *are both aggregation functions;*
- *both* $\triangledown$ *and* $\vartriangle$ *are commutative binary operations;*
- *both* $\triangledown$ *and* $\vartriangle$ *are associative binary operations; and*
- $\mathsf{A} \triangledown \mathsf{B} \geq \max\{\mathsf{A}, \mathsf{B}\}$ *and* $\mathsf{A} \vartriangle \mathsf{B} \leq \min\{\mathsf{A}, \mathsf{B}\}$.

Example 4.8. Let $\boldsymbol{\alpha} \in [0,\infty[^n$ be a vector of non-negative coefficients, and let $\mathsf{W}_{\boldsymbol{\alpha}}\colon [0,\infty[^n \to [0,\infty[$ be the weighted average on $[0,\infty[^n$ with respect to the vector $\boldsymbol{\alpha}$, i.e.,

$$\mathsf{W}_{\boldsymbol{\alpha}}(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i x_i,$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\mathbf{x} = (x_1, \dots, x_n)$. Then, one, after an easy computation (see [16]), can obtain that $\mathsf{W}_{\boldsymbol{\alpha}} \triangledown \mathsf{W}_{\boldsymbol{\beta}} = \mathsf{W}_{\boldsymbol{\alpha} \vee \boldsymbol{\beta}}$ and $\mathsf{W}_{\boldsymbol{\alpha}} \vartriangle \mathsf{W}_{\boldsymbol{\beta}} = \mathsf{W}_{\boldsymbol{\alpha} \wedge \boldsymbol{\beta}}$.

To increase the optimality of dividing resources $\mathbf{x}$ between the two consumers we may consider also 'inter-division' for one consumer similarly as in the case of super- and sub-additive transformations. We then obtain the following convolutions.

Definition 4.6. Let $\mathsf{A}\colon [0,\infty[^n \to [0,\infty[$ and $\mathsf{B}\colon [0,\infty[^n \to [0,\infty[$ be two aggregation functions. A *super-convolution* of these aggregation functions is a mapping $\mathsf{A} \circledtriangledown \mathsf{B}\colon [0,\infty[^n \to [0,\infty[$ such that

$$\mathbf{x} \mapsto \sup \left\{ \sum_{i=1}^{m} \mathsf{A}(\mathbf{y}_i) + \sum_{i=1}^{k} \mathsf{B}(\mathbf{z}_i)\colon \mathbf{y}_i, \mathbf{z}_i \in [0,\infty[^n, \sum_{i=1}^{m} \mathbf{y}_i + \sum_{i=1}^{k} \mathbf{z}_i = \mathbf{x}, m, k \in \mathbb{N} \right\}.$$

A *sub-convolution* is a mapping $\mathsf{A} \circledvartriangle \mathsf{B}\colon [0,\infty[^n \to [0,\infty[$ such that

$$\mathbf{x} \mapsto \inf \left\{ \sum_{i=1}^{m} \mathsf{A}(\mathbf{y}_i) + \sum_{i=1}^{k} \mathsf{B}(\mathbf{z}_i)\colon \mathbf{y}_i, \mathbf{z}_i \in [0,\infty[^n, \sum_{i=1}^{m} \mathbf{y}_i + \sum_{i=1}^{k} \mathbf{z}_i = \mathbf{x}, m, k \in \mathbb{N} \right\}.$$

Unfortunately, it may happen that $\mathsf{A} \circledtriangledown \mathsf{B}$ is not well-defined, i.e., there exists $\mathbf{x}$ such that $(\mathsf{A} \circledtriangledown \mathsf{B})(\mathbf{x}) = \infty$. In [16] it is shown that this occurs if and only if A^* or B^* escapes. In the following, we will assume that $\mathsf{A} \circledtriangledown \mathsf{B}$ is well-defined without explicitly saying so.

The following theorem sums up the properties of super- and sub-convolutions. Also their relation with convolution and lower convolution is given.

Theorem 4.16 (Propositions 3.6, 3.7, 3.8 and 3.9 in [16])**.** *Let* $\mathsf{A}, \mathsf{B} \in \mathbb{A}$ *be two aggregation functions. Then*

- $\mathsf{A} \circledtriangledown \mathsf{B}$ *is an aggregation function if and only if* A^* *and* B^* *do not escape;*
- $\mathsf{A} \circledtriangleup \mathsf{B}$ *is an aggregation function;*
- *both* $\circledtriangledown$ *and* $\circledtriangleup$ *are commutative binary operations;*
- *both* $\circledtriangledown$ *and* $\circledtriangleup$ *are associative binary operations; and*
- $\mathsf{A} \circledtriangledown \mathsf{B} \geq \max\{\mathsf{A}^*, \mathsf{B}^*\}$ *and* $\mathsf{A} \circledtriangleup \mathsf{B} \leq \min\{\mathsf{A}_*, \mathsf{B}_*\}$.

A topic of super- and sub-additive transformations of convolutions was also included in [16]. The results are summarized in the following theorem.

Theorem 4.17 (Theorems 4.1 and 4.4 in [16])**.** *Let* $\mathsf{A}, \mathsf{B} \in \mathbb{A}$ *be two aggregation functions. Then* $(\mathsf{A} \triangledown \mathsf{B})^* = \mathsf{A}^* \triangledown \mathsf{B}^* = \mathsf{A} \circledtriangledown \mathsf{B}$ *and* $(\mathsf{A} \vartriangle \mathsf{B})_* = \mathsf{A}_* \vartriangle \mathsf{B}_* = \mathsf{A} \circledtriangleup \mathsf{B}$.

Also the problem of self-convolution was solved and is presented in the following theorem.

Theorem 4.18 (Theorem 5.4 in [16])**.** *Let* $\mathsf{A} \in \mathbb{A}$ *be any aggregation function. Then* $\mathsf{A} \triangledown \mathsf{A} = \mathsf{A}$ *(*$\mathsf{A} \vartriangle \mathsf{A} = \mathsf{A}$*) if and only if* A *is super-additive (sub-additive).*

Chapter 5

Concluding remarks and further development of the topic

In this dissertation thesis, we have expanded the theory of decomposition integrals and also the theory of aggregation functions. In Chapter 3, we found a few construction methods of decomposition integrals that extend the Lebesgue integral and found sufficient and necessary conditions for decomposition integrals to be the extension of the Lebesgue integral. Then, we extended the framework of decomposition integrals for interval-valued functions. We considered two different extensions, the first one was based on the Aumann integral, i.e., based on creating an envelope of all possible values of decomposition integrals of selectors of a given interval-valued function. The second is based on a direct modification of the definition of decomposition integrals using interval algebra. Interestingly, both approaches lead to the same operator and are thus equivalent. Also, we introduced and paid our attention to special decomposition integrals called collection integrals; in this case the decomposition system consists of a single collection. We have characterized all collection integrals that can be represented by the Choquet integral and those that obey some of the integral inequalities, including Jensen's, Chebyshev's, Cauchy's, and Hölder's integral inequalities. We have also found an interesting application of the collection integrals in the theory of imprecise probabilities, and we have used the integrals to construct coherent upper and lower previsions. Decomposition integrals are hard to compute, in general, and thus we introduced a modification of decomposition integrals, called greedy decomposition integrals, that are much easier to compute. Unfortunately, greedy decomposition integrals lack some 'good' properties such as monotonicity. We have concluded the chapter with a discussion about the computational complexity of computing some specific decomposition integrals.

In the following chapter, Chapter 4, we summarized our results for the theory of aggregation functions. We defined new transformations of aggregation functions and discussed their relationship with the super- and sub-additive transformations of aggregation functions. We were also interested in the problem of preserving different types of continuities after applying super- and sub-additive transformations to aggregation functions. Lastly, we have defined four different ways to convolute aggregation functions and examined their properties.

In the future, we would like to:

- extend some results on collection integrals for all decomposition integrals, i.e., characterize all decomposition integrals that can be represented by the Choquet integral;
- provide the full characterization of those decomposition integrals and decomposition systems that extend the Lebesgue integral;
- find more applications of decomposition integrals in other parts of mathematics, e.g., in image processing or multi-criteria decision making;
- modify the definition of decomposition integrals by replacing the standard summation and multiplication by other operations on $[0, \infty]$;
- continue to solve the problem of continuity preservation for different transformations of aggregation functions.

Chapter 6

Author's work related to Dissertation thesis

- Doria, S., Mesiar, R., Šeliga, A.: Integral representation of coherent lower previsions by super-additive integrals. Axioms **9**(2) (2020), art.no. 43.
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- Li, J., Mesiar, R., Ouyang, Y., Šeliga, A.: Characterization of decomposition integrals extending Lebesgue integral. Accepted for publication in *Fuzzy Sets and Systems*.
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