**Slovak University of Technology in Bratislava**
**Faculty of Civil Engineering**

**Ing. Andrea Ábrahámová**

Dissertation Thesis Abstract

# Mathematical transformation methods of data in geodesy and cartography

to obtain the Academic Title of *Philosophiae doctor* (PhD.)

in the doctorate degree study program

9.1.9 Applied Mathematics

full-time study

Bratislava 2022

The dissertation thesis has been prepared at the Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava.


**Submitter:** Ing. Andrea Ábrahámová
Department of Mathematics and Descriptive Geometry
Faculty of Civil Engineering, STU, Bratislava

**Supervisor:** doc. RNDr. Margita Vajsáblová, PhD.
Department of Mathematics and Descriptive Geometry
Faculty of Civil Engineering, STU, Bratislava

**Readers:** prof. Ing. Juraj Janák, PhD.
Department of Theoretical Geodesy and Geoinformatics
Faculty of Civil Engineering, STU, Bratislava

doc. RNDr. Daniela Velichová, CSc.
Institute of Mathematics and Physics
Faculty of Mechanical Engineering, STU, Bratislava

Ing. Daniel Szatmári, PhD.
Institute of Geography
Slovak Academy of Sciences


Dissertation Thesis Abstract was sent: ……………………………………

Dissertation Thesis Defence will be held on …………………… at ……… am/pm at the Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Radlinského 11, 810 05 Bratislava

Prof. Ing. Stanislav Unčík
Dean of Faculty of Civil Engineering

**Contents**



## Abstrakt

Zber dát je v súčasnosti realizovaný najmodernejšími technológiami, ako LiDAR, GNSS, čo si vyžaduje použitie efektívnych matematických metód na ich spracovanie, analýzu až po zobrazenie v kartografickom zobrazení. Dizertačná práca sa zameriava na použitie matematických metód vo fáze kartografického zobrazovania a na optimalizáciu dosiahnutých skreslení. Aplikácia alternatívnych matematických metód pri tvorbe kartografických zobrazení je zaujímavým faktorom, ktorý ovplyvňuje optimalizáciu hodnôt skreslenia, ako aj ich rozloženie na ploche zobrazovaného územia. Táto práca prezentuje metodiku tvorby a porovnanie skreslení v kartografických zobrazeniach tvorených alternatívnymi matematickými metódami na Slovensku, predovšetkým riešením Laplaceovej alebo Poissonovej rovnice, ktoré vedú k minimalizácii integrálneho, resp. súčtového kritéria pre moduly

dĺžkového skreslenia. V práci je prezentovaná tvorba variačného konformného zobrazenia, minimaximálneho konformného zobrazenia a konformného kartografického zobrazenia na základe riešenia Poissonovej rovnice Ritzovou metódou, ktoré boli následne porovnané. Optimalizácia na základe riešenia Laplaceovej rovnice metódou konečných prvkov bola aplikovaná na štyri navrhované kartografické zobrazenia, a to na Lambertovho konformné kužeľové zobrazenie v normálnej polohe, konformné kužeľové zobrazenie v normálnej polohe s minimálnou strednou kvadratickou hodnotou dĺžkového skreslenia v normálnej polohe, konformné kužeľové zobrazenie vo všeobecnej polohe a konformné kužeľové zobrazenie s minimálnou strednou kvadratickou hodnotou dĺžkového skreslenia vo všeobecnej polohe. Tvorba variačného a minimaximálneho kartografického zobrazenia je založená na Airyho-Kavrajskom hodnotiacom kritériu, čo vedie k riešeniu Laplaceovej rovnice. Kartografické zobrazenie vytvorené Ritzovou metódou spĺňa Chebyshevovo kritérium, ktoré možno matematicky definovať Poissonovou rovnicou. Zobrazovacie rovnice uvedených kartografických zobrazení vychádzajú z Taylorovho radu za podmienky konformity ich zhody v komplexnej rovine. Metóda konečných prvkov sa ukazuje ako veľmi vhodná výpočtová metóda na riešenie Laplaceovej rovnice vyjadrujúcej geodetické a iné problémy nielen v inžinierskej praxi.
**Kľúčové slová:** konformné zobrazenie, dĺžkové skreslenie, Laplaceova rovnica, Poissonova rovnica, variačné kritérium, Ritzova metóda, Metóda konečných prvkov

## Introduction

The application of alternative mathematical methods in creating cartographic projections is an interesting factor that affects the optimization of distortion values, as well as their distribution in the area of the projected territory. This thesis presents the creation methodology and comparison of distortions in cartographic projections formed by alternative mathematical methods of minimizing the integral, resp. sum criterion for scale distortion in Slovakia, namely, the creation of the variational conformal projection, minimaximal conformal projection, and the conformal cartographic projection created by the Ritz method, and their optimizations based on solving the Laplace equation by the Finite Element Method. The modification was applied to four proposed cartographic projections, namely, Lambert's conformal conic projection in a polar aspect, Conformal conic projection in a polar aspect with minimizing Root Mean Square of the scale distortion values, Conformal conic projection in an oblique aspect, and Conformal conic projection in an oblique aspect with minimizing RMS of the scale distortion values. The creation of the variational and minimaximal cartographic projection is based on the Airy-Kavraiskii criterion of evaluating the projection on the displayed area by solving the Laplace equation. The cartographic projection created by the Ritz method satisfies the criterion of Chebyshev, which can be defined mathematically by the Poisson equation. The Finite Element Method proves to be a very suitable computational method for solving the Laplace equation expressing geodetic and other problems not only in engineering practice. The map equations of the mentioned cartographic projections are based on the Taylor Series of conditions for their conformity in the complex plane.

## 1 Foundation of mathematical cartography

One of the main tasks of mathematical cartography is to determine a projection of a projected territory in such a way that the resulting deformations of the original map elements are objectively minimized.

Earth's exact shape is complex, and to project this shape, the complexity needs to be simplified by geometric surfaces, so-called reference surfaces. Mathematical cartography generally uses three different reference surfaces: **plane**, **sphere**, and **ellipsoid**. With an **ellipsoid**, a unique radius value for its equatorial and polar axes is specified, which approximates the overall Earth's shape better than with a sphere. Its surface is used to approximate the actual surface of Earth for geodetic positioning, and subsequently, for geodetic mapping. The basic parameters of an ellipsoid include the semi-major axis $a$, the semi-minor axis $b$, and the first numerical eccentricity $e$. The position of the points on the reference ellipsoid is uniquely determined by their ellipsoidal latitude $\varphi$ and ellipsoidal longitude $\lambda$. The simplest way to think of the Earth's shape is to model it as a **reference sphere**, which is a special case of surfaces of revolution because it has a constant radius $R$. The position of the points on the reference sphere is uniquely determined by their spherical latitude $U$ and spherical longitude $V$. In the **projection plane**, we most often work with a rectangular coordinate system with the beginning in the point $O$, and the position of the points is expressed with the Cartesian coordinates $x$ and $y$. In the plane also appear polar coordinates, the polar radius $\rho$, and the polar angle $\varepsilon$.

The isometric coordinates are extremely important in cartography since the determination of these coordinates on surfaces leads directly to conformal mappings, probably the most important type of projection from a practical, as well as a theoretical, point of view. Isometric coordinates on the reference ellipsoid are the isometric longitude $\lambda$, which is equal to the ellipsoidal longitude, and the ellipsoidal isometric latitude $q$, which is given by the formula:

$$q = \ln\left[\tan\left(\frac{\varphi}{2}+\frac{\pi}{4}\right)\sqrt{\left(\frac{1-e\sin\varphi}{1+e\sin\varphi}\right)^{e}}\right], \tag{1.1}$$

where $e$ is the first numerical eccentricity for the reference ellipsoid.

Isometric coordinates on the reference sphere are the isometric longitude $V$, which is equal to the spherical longitude, and the spherical isometric latitude $Q$, which is obtained from the formula (1. 9) if $e = 0$:

$$Q = \ln\left[\tan\left(\frac{U}{2}+\frac{\pi}{4}\right)\right]. \tag{1.2}$$

Differential geometry (Goetz, 1970) shows that an isometric mapping of two surfaces, where all corresponding distances on both surfaces remain identical, can be obtained if and only if their Gaussian curvatures are identical. The transformation and the mapping process will always cause distortions of the original elements. However, some elements may be preserved in the transformation process, and the complete identity of the reference surface elements and their projected equivalents can never be achieved in cartographic projections (Frankich, 1982).

The **distortion of distances (scales)** in the cartographic projection is characterized by the scale distortion factor, which is defined:

$$m = \frac{\mathrm{d}S}{\mathrm{d}s}, \tag{1.3}$$

where d$S$ is the geodetic element in the plane of the projection, and d$s$ is its corresponding element on the reference surface.

The **angular distortion** is defined as the difference between the image of the angle $\omega$ in the projection plane and the corresponding angle $\omega$ on the reference surface:

$$\Delta\omega = \omega' - \omega. \tag{1.4}$$

The **areal distortion** is characterized in the cartographic projections by the areal distortion factor $m_{pl}$, which is defined as the ratio of the areal element d$P$ on the projection plane to the corresponding areal element d$p$ on the reference area:

$$m_{pl} = \frac{\mathrm{d}P}{\mathrm{d}p}. \tag{1.5}$$

The **conformal projections** are the most frequently applied map projections in the geodetic coordinate systems. In the conformal projections, the angles, i.e., the azimuths and the bearings, are preserved. A necessary and sufficient condition of the conformal projections is the equality of the scale distortion in the direction of the parallels $m_p$ and the meridians $m_r$, and zero distortion of the right angle of the meridians and parallels (Hoffmann, Hojovec, 1976).

From the general relations for the cartographic projections (Urmajev, 1947) we get the so-called **Laplace equation for the conformal projections**:

$$\frac{\partial^2 \ln\mu}{\partial Q^2} + \frac{\partial^2 \ln\mu}{\partial V^2} = 0, \tag{1.6}$$

thus a second-order differential equation in which $Q$ is the spherical isometric latitude (1.2) and $\mu$ is:

$$\mu = m\cos U = m\operatorname{sech} Q, \tag{1.7}$$

where $m$ is the scale distortion factor and $U$ is the spherical latitude.

According to (1.7) ln($\mu$) is equal to:

$$\ln\mu = \ln m + \ln\cos U \text{ or } \ln\mu = \ln m - \ln\cosh Q\,\cdot \tag{1.8}$$

Substituting the expression (1.8) into the equation (1.6) we can convert the Laplace equation to **the Poisson equation**:

$$\frac{\partial^2 \ln m}{\partial Q^2} + \frac{\partial^2 \ln m}{\partial V^2} = f, \tag{1.9}$$

where the right-hand-side $f$ equals $f = \operatorname{sech}^2 Q$.

The **evaluation of the cartographic projection** is used for the selection and creation of the most advantageous projection for a given territory, but also the applied projections are evaluated. The evaluation criteria are divided into two types (Hojovec et al., 1987):

- extreme and minimaximal,
- variational (sum and integral criteria).

In the thesis, the most frequently used variational criterion is the **Airy – Kavraiskii criterion**, which is formulated in the form:

$$h^2 = \frac{1}{2}\left[\ln^2(m_a) + \ln^2(m_b)\right], \tag{1.10}$$

where in the conformal projections deals their equality $m = m_a = m_b$ as follows:

$$h^2 = \ln^2 m. \tag{1.11}$$

By applying the previous variational criteria at the point, we get a characteristic value for the whole displayed territory Ω of the reference surface, namely the so-called **integral variational criterion**:

$$I^2 = \frac{1}{p} \iint_{\Omega} h^2 \, \mathrm{d}\Omega, \tag{1.12}$$

and $p$ is the area value of the domain Ω, and $m$ is the scale distortion factor in a map projection. For $n$ points evenly distributed through the territory the **sum variational criterion** applies:

$$I^2 = \frac{1}{n} \sum_{i=1}^{n} h^2. \tag{1.13}$$

## 2 Objectives of the dissertation thesis

The aim of the presented dissertation thesis is the methodology of data transformation in geodesy and cartography computed by innovative mathematical methods, to find the optimal method for minimizing distortion according to the established criteria, to optimize the analysis of distortions in conformal cartographic projections for the territory of Slovakia by solving the Laplace equation or the Poisson equation by various numerical methods e.g.:

- Method of Least Squares, which leads to variational and minimaximal projections,
- the Ritz method,
- Finite Difference Method (FDM),

- Finite Element Method (FEM),

and comparison with projections in a given class.

## 3 The optimization criteria in the creation of the conformal projections

Selecting a suitable map projection for a mapping project is probably the most important and challenging aspect of working with projections. A modification of a cartographic projection is the process of obtaining new projections from already existing projections (Kessler, Battersby, 2019).

Most solution methods of the Dirichlet problem in the mathematical cartography can be subdivided into two fundamentally different groups. There are suggested solutions of various methods to the optimization, e.g. the numerical methods, like the Ritz method, Method of Finite Differences, Method of Finite Elements, and the Method of Least Squares, which are based on the theorem of Chebyshev and Airy-Kavraiskii and lead us to the variational and minimaximal type of projections. The first group of the solutions, in which the result is an approximate analytic function, i.e. an approximation of the harmonic function that perfectly satisfies the boundary conditions. The best known among these methods is the Ritz method, which was developed in 1908 by the German engineer Walther Ritz. The second group of methods yields a rigorous harmonic function that does not perfectly satisfy the boundary condition. The most suitable method from the second group is the Method of Least Squares, which leads to the variational and minimaximal type of cartographic projections. The boundary of the domain is usually approximated by a series of discrete points, the boundary will be a closed polygon and the line of constant scale distortion will be a smooth curve that approximates a polygon.

Since the transformation will generally change the original proportions, it is important to adopt the scale distortions as the basic parameter for the evaluation of the cartographic projections. As a qualitative measure of the map projections, we decided to use the **Airy-Kavraiskii criterion** (1.13).

The solution of the conformal projections consists of solving the Laplace equation or solving the Poisson equation with zero boundary conditions if for the searched function $\ln \mu$ the following will apply at the boundary of the given area:

$$\ln \mu = u(Q,V) = \ln \operatorname{sech} Q \cdot \tag{3.1}$$

The solution to this problem, called the Dirichlet problem, is a harmonic function. Finding such a harmonic function that exactly meets the required conditions from a mathematical point of view is difficult. Therefore, approximation solutions are used. One aspect to solve it is to use the so-called homogeneous harmonic polynomials derived from the function of a complex variable.

The projection area for the optimization of the conformal cartographic projections was defined in various ways, such as a spherical geographic trapezoid, a spherical cartographic trapezoid, and an area using the real border of the country's territory.

### 3.1 Variational cartographic projections

The creation of the variational cartographic projection is based on the Airy-Kavraiskii criterion for evaluating a projection in the projected area. In the variational projection, we consider its symmetry and the relative spherical latitudes $V$, resp. the relative cartographic latitudes $D$, determined from the standard spherical or cartographic meridian The basic principle of the variational projection is to find such a function ln($m$) that minimizes the value of criterion $I$, which is satisfied in the conformal projection if the Laplace equation applies (1.6). where function $u$ is:

$$u = \sum_{j=0}^{n} \left( a_j \psi_j + b_j \tau_j \right), \tag{3.2}$$

where $\psi_j$ and $\tau_j$ are defined as (Hojovec, 1996):

$$\psi_j + i\tau_j = (Q + iD)^j. \tag{3.3}$$

Two complex variables are equal if their real and imaginary components are equal. Then for $\psi_j$ and $\tau_j$, when choosing the value $j$ e.g. $j = 0$ to $j = 6$ we obtain:

$$\begin{aligned}
&\psi_0 = 1,\ \psi_1 = Q,\ \psi_2 = Q^2 - D^2,\ \psi_3 = Q^3 - 3QD^2,\ \psi_4 = Q^4 - 6Q^2D^2 + D^4, \\
&\psi_5 = Q^5 - 10Q^3D^2 + 5QD^4, \psi_6 = Q^6 - 15Q^4D^2 + 15Q^2D^4 - D^6, \\
&\tau_0 = 0,\ \tau_1 = D,\ \tau_2 = 2QD,\ \tau_3 = 3Q^2D - D^3,\ \tau_4 = 4Q^3D - 4QD^3, \\
&\tau_5 = 5Q^4D - 10Q^2D^3 + D^5, \tau_6 = 6Q^5D - 20Q^3D^3 + 6QD^5.
\end{aligned} \tag{3.4}$$

The solution of the Laplace equation (1.6) for a symmetric territory is simplified:

$$u = \sum_{j=0}^{n} a_j \psi_j = \ln(m \cos S) = \ln m + \ln \cos S, \tag{3.5}$$

from where we express ln($m$):

$$\ln m = \sum_{j=0}^{n} a_j \psi_j - \ln \cos S. \tag{3.6}$$

From the system of $n$ + 1 equations, we get the values of the coefficients $a_0$, $a_1$, ..., $a_n$ for the area determined by the extreme parallels $S_{\min}$, $S_{\max}$, and the meridians 0°, $\Delta D_{\max}$. We calculate the scale distortion factor $m$ from the relation (3.6).

For a symmetric territory, where the solution of the Laplace equation is a function $u$ according to (3.5), for the middle meridian $D_0$, where $\Delta D$ = 0°, the following holds:

$$\ln v = \sum_{j=0}^{n} a_j Q^j . \tag{3.7}$$

In the derivation of map equations, we use relations (4) based on the notation of a conformal projection in a complex plane according to Gauss.

### 3.2 Minimaximal cartographic projections

We can talk about the minimaximal type of projection if one of the minimaximal criteria, such as the Chebyshev theorem, is used in deriving the projections. The aim is to find such a conformal projection for a given area that the values of the scale distortion factors on a closed boundary curve should be a constant value. The design of the minimaximal projection consists of the calculation of projection's coefficients $a_j$, $b_j$ from the condition that the values of the scale distortion factors on the closed boundary curve differ as little as possible from zero.

Subsequently, from the relation (3.6) we calculate the values of the scale distortion factors $m$ at the boundary, and multiply them by a suitable scale distortion factor $m_s$ that the scale distortion values satisfy the condition:

$$\frac{|m_{\max} - 1|}{|m_{\min} - 1|} = 1 \cdot \tag{3.8}$$

Then it is true that the undistorted isometric line is not at the boundary of the territory, but inside the territory, the values of extreme-scale distortions will be uniform and meet the minimaximal criterion.

### 3.3 The Ritz method for solving the Poisson equation

The Ritz method, which was developed in 1908 by the German engineer Walther Ritz, is one of the various ways of minimizing the Dirichlet integral. This is a direct method and relies on the approximate solution for boundary value problems; in our case, especially it is an approximate solution of the Poisson equations with zero BC (1.9), where the right-hand side of equation $f$ equals $f = \operatorname{sech}^2(Q)$.

The solution will provide us with a function $u$ ($Q$, $V$) which is continuous in the domain $\Omega$, together with its partial derivatives of the first and second orders and it vanishes along the boundary $\partial\Omega$ (Frankich, 1982). In the literature (Frankich, 1982) is the methodology of the calculation of the cartographic projections using the Poisson equation solved by the Ritz method for the first equation from a family of functions $u$ described. In this thesis, we formulate the calculation for the 4$^{th}$ equation of the family of functions $u$, which contains more coefficients, and clarify the results of the scale distortion values.

### 3.4. Numerical solutions of the Partial Differential Equations for the modification of cartographic projections

The methods described above are not the only methods of finding a function $u$, as an approximate solution of the Laplace equation or the Poisson equation with zero boundary condition (BC). To solve Partial Differential Equations (1.6) and (1.9) the Finite Difference Method (FDM) and the Finite Elements Method (FEM) were used.

Our goal was to obtain the scale distortion factor $m$ within the given domain $\Omega$ based on solving the Laplace equation (LE) for the conformal projections (1.6), while at the boundary of the given domain $\Omega$ we prescribe the values of the scale distortion $m$ (Ábrahámová, 2020).

## 4 Application of the solving of the Laplace and the Poisson equations in optimization of the conformal projections in the territory of the Slovak Republic

The focus of this thesis is the selection of a conformal cartographic projection for the territory of the Slovak Republic in terms of distortion values, using one of the mentioned, and the described optimization criteria in conformal projections.

As input data, we used the ellipsoidal latitude $\varphi$ and ellipsoidal longitude $\lambda$ for 2,366 points of the border of the Slovak Republic on the ellipsoid GRS80. Transformation of the ellipsoidal coordinates $\varphi$, $\lambda$ into the spherical coordinates $U$, $V$ is necessary before projecting all types of used conformal projections.
For the transformation, we used the Gaussian conformal projection with the following parameters, which apply to the Slovak Republic: $k = 1.003315995637$, $\alpha = 1.000640596751$, and radius of the reference sphere $R = 6{,}380{,}840.721$ m. The standard parallel is $\varphi_0 = 48°\ 40'\ 21.2520''$, and the standard meridian is $\lambda_0 = 19°\ 41'\ 56.0940''$. The equations for the calculation of the parameters and the map equation of the Gaussian conformal projection of the ellipsoid to the sphere are given in (Vajsáblová, 2021):

$$\tan\left(\frac{U}{2} + 45°\right) = k\left[\tan\left(\frac{\varphi}{2} + 45°\right)\sqrt{\left(\frac{1 - e\sin\varphi}{1 + e\sin\varphi}\right)^{e}}\right]^{\alpha}, \qquad V = \alpha\,\lambda. \tag{4.1}$$

In the case of a spherical cartographic trapezoid, we need to introduce another step of the calculation, the transformation of the spherical geographic coordinates $U$ and $V$ into cartographic coordinates $S$ and $D$ with the equations (Vajsáblová, 2021):

$$\sin S = \sin U \sin U_K + \cos U \cos U_K \cos(V_K - V), \qquad \sin D = \frac{\cos U}{\cos S}\sin(V_K - V), \tag{4.2}$$

where for the territory of the Slovak Republic, the position of the cartographic pole was proposed as follows (Vajsáblová, 2015): $U_K$ = -5° 53′ 41.1964″, $V_K$ = 32° 8′ 18.5219″.

### 4.1 Variational projection, minimaximal projection, and the map projection with the Ritz method, and their comparison in the Slovak Republic

The cartographic projections presented in this part of the section have in common that they are based on the solution of Laplace, reps. the Poisson equation and the sought functions of the map equations satisfy the Gaussian notation of conformal projections in the complex plane.

We can define the computation area for the territory of the Slovak Republic in various ways. Next, we will focus on the proposals of the spherical trapezoid in the oblique aspect, the so-called **spherical cartographic trapezoid**.

In the **variational projection**, our domain, i.e., the **spherical cartographic trapezoid** bounded by the cartographic parallels with the cartographic latitudes $S_{\max}$, $S_{\min}$ and the cartographic meridians with the cartographic longitude $D_{\max}$, $D_{\min}$, was discretized by the dividing step 1′, then, together with the discrete points of the real border of the Slovak Republic, we obtained $n$ = 28,560 discrete points. For the spherical cartographic trapezoid thus defined, the values of coefficients $a_j$ are calculated ($n$ = 6) as follows:

$$a_0 = 1.278069,\ a_1 = -2.899471,\ a_2 = -7.334089,\ a_3 = 17.690263,\ a_4 = 14.447583,$$
$$a_5 = -55.201458,\ a_6 = 32.819466.$$

Scale distortion was calculated according to formula (3.6), and its values range from -3.97 cm/km to 7.13 cm/km. Finally, the values of the Airy-Kavraiskii sum criterion were calculated on a dense set of points evenly distributed on the territory of Slovakia based on equation (1.13), where $h$ is from equation (1.11), and its value is 2.55 cm/km. The isometric lines of the scale distortion factor $m$ in the considered number of polynomials $n$ = 6 are shown in Fig. 4.1. In Fig. 4.2, we can see the Cartesian coordinates $x$ and $y$ for the spherical cartographic coordinates.

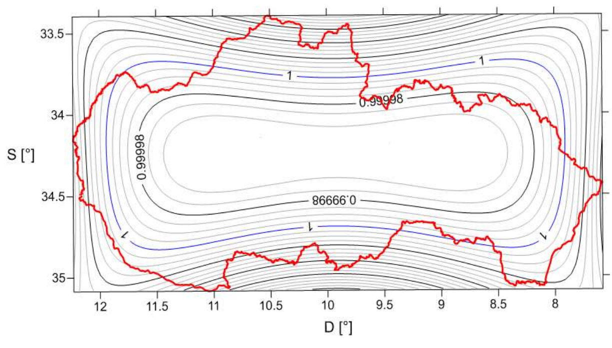


**Fig. 4.1** The isometric lines of the scale distortion factor $m$ for the spherical cartographic trapezoid ($n$ = 6)

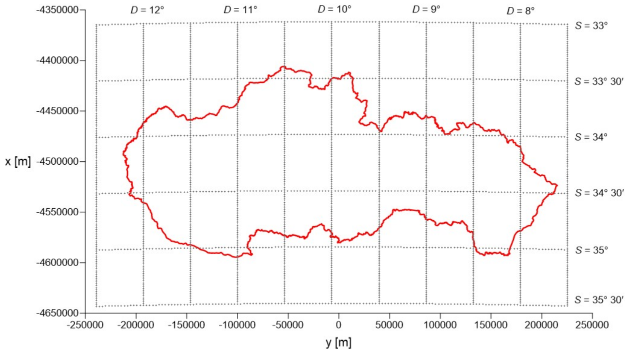


**Fig. 4.2** Cartesian coordinates $x$, $y$ of the Slovak Republic and the cartographic network in the variational projection for the spherical cartographic trapezoid

In the **minimaximal projection**, our domain, the spherical cartographic trapezoid was defined by $n = 764$ points at the boundary of the calculation area using a constant step 1′ on the cartographic parallel and the meridian. The coefficient values were calculated for $n = 6$, and are equal to:

$$a_0 = 0.814517,\ a_1 = 3.723066,\ a_2 = -46.617546,\ a_3 = 136.932128,\ a_4 = -180.325234,$$
$$a_5 = 108.062032,\ a_6 = -22.416667.$$

The scale distortion factor $m$ was calculated according to the formula (3.6) for the set of points discretized by step 1′ in the territory of Slovakia ($n = 28{,}560$ points) and multiplied by the scale factor $m_{sm}$, and its values range from -5.9 cm/km to 5.9 cm/km. Finally, the values of the Airy-Kavraiskii sum criterion were calculated from equation (1.13), and its value is 3.23 cm/km. The isometric lines of the scale distortion factor $m$ at the considered number of polynomials $n = 6$ are shown in Fig. 4.3. In Fig. 4.4, we can see the Cartesian coordinates $x$ and $y$ for the spherical cartographic coordinates.

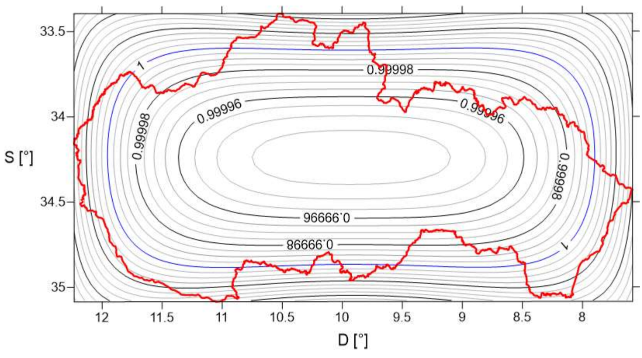


**Fig. 4.3** The isometric lines of the scale distortion factor $m$ for the spherical cartographic trapezoid for the minimaximal projection ($n = 6$)

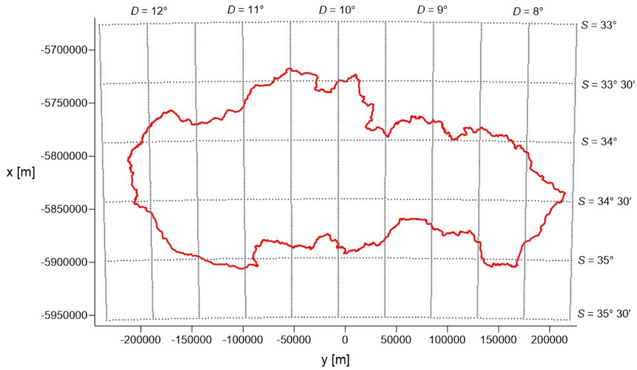


**Fig. 4.4** Cartesian coordinates $x$, $y$ of the Slovak Republic and the cartographic network in the minimaximal projection for the spherical cartographic trapezoid

In the application of the **cartographic projection using the Poisson equation solved by the Ritz method**, the fourth equation of the family of functions $u$ is applied for the spherical cartographic trapezoid and also for the spherical cartographic trapezoid with the following substitution in equations $V = D$, $\Delta V = \Delta D$:

$$u = \left(a^2 - \Delta Q^2\right)\left(b^2 - \Delta V^2\right)\left(c_1 + c_2 \Delta Q + c_3 \Delta V^2 + c_4 \Delta Q^3\right) \tag{4.3}$$

Our domain, i.e. the territory of the Slovak Republic, is bounded by a spherical cartographic trapezoid in an oblique aspect. The mapping domain was discretized by step 1' ($n$ = 28,560 discrete points). Since the spherical isometric latitudes $Q$ appear in the equations for the calculation of the projection, the spherical cartographic latitudes $S$ were transformed into the spherical isometric latitudes $Q$ according to (1.2).

The final values of the coefficients $c_i$ are as follows:

$c_1$ = -217.2715, $c_2$ = 92.9703 $c_3$ = 351879.3272, $c_4$ = -290512.7582.

The scale distortion factor $m$ was calculated on the relation $u = \ln(m)$. The resulting scale distortions range from -13.26 cm/km to 0 cm/km. The values of the Airy-Kavraiskii sum criterion were calculated from equation (3.2), and its value is 8.24 cm/km. The isometric lines of the scale distortion factor $m$ in the cartographic projection using the PE solved by the Ritz method are shown in Fig. 4.5. In Fig. 4.6, we can see the Cartesian coordinates $x$ and $y$ for the spherical cartographic coordinates.

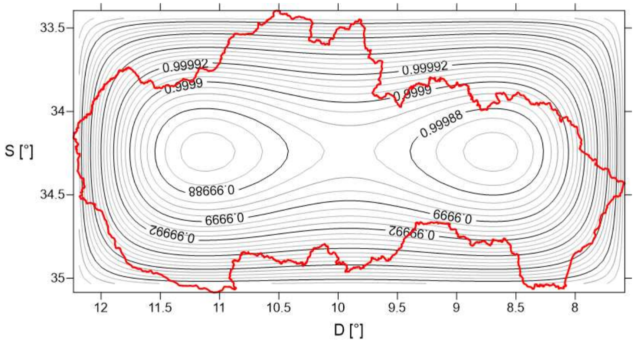


**Fig. 4.5** The isometric lines of the scale distortion factor $m$ in the cartographic projection using the PE solved by the Ritz method for the spherical cartographic trapezoid

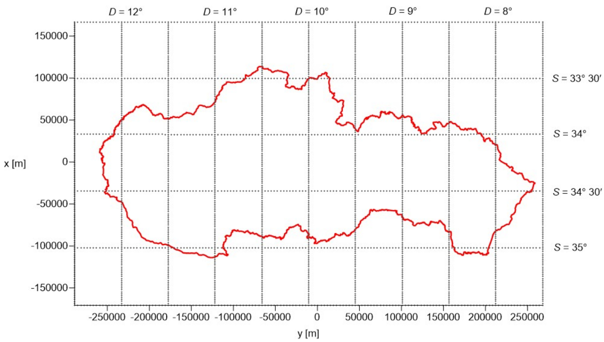


**Fig. 4.6** Cartesian coordinates $x$, $y$ of the cartographic projection using the PE solved by the Ritz method in the Slovak Republic for the spherical cartographic trapezoid

From the results, we can see that the boundary conditions were preserved, because the boundary of the domain Ω, the Slovak Republic, which is bounded by a spherical geographic, resp. a spherical cartographic trapezoid is not distorted. To obtain more harmonical distortion, we decided to define boundary conditions inside the computational area. The zero BC was moved inside the spherical cartographic trapezoid by 17′, then the limit values of the boundary conditions are as follows:

- cartographic latitude: $S_1 = 33° \, 40' \, 42.1307''$, $S_2 = 34° \, 48' \, 5.7509''$,
- cartographic longitude: $D_1 = 7° \, 51' \, 55.6498''$, $D_2 = 11° \, 56' \, 59.6088''$.

The final values of the coefficients $c_i$ were calculated from the equation (3.50), their values are as follows:

$$c_1 = -302.6927, \; c_2 = 122.8710, \; c_3 = 1041150, \; c_4 = -6422897.$$

The function $u$ was calculated from the equation (4.3), and the scale distortion factor $m$ was calculated based on the relation $u = \ln(m)$, and was subsequently quantified in cm/km.

By the used dividing step ($n$ = 28,560), the resulting scale distortions are based from -6.85 cm/km to 8.62 cm/km. The isometric lines of the scale distortion factor $m$ of the cartographic projection using the PE solved by the Ritz method are shown in Fig. 4.7. Finally, the value of the Airy-Kavraiskii sum criterion was calculated on a dense set of points evenly distributed on the territory of Slovakia according to equation (1.13) and its value is 4.13 cm/km

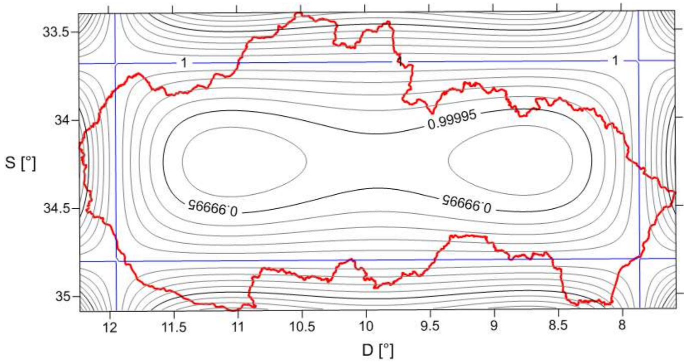


**Fig. 4.7** Isometric lines of the scale distortion factor $m$ in the cartographic projection using the PE solved by the Ritz method for the modified spherical cartographic trapezoid

In the cartographic projections analyzed in normal and oblique aspects, we calculated the characteristics of the scale distortions listed in Tab. 4.1. All of the projections are conformal, so they preserve the angles. The above analysis showed that the extreme values of the scale distortions are much smaller than the criterion of ±10 cm/km required by the Geodesy, Cartography and Cadastre Authority of the Slovak Republic.

Fig. 4.8 illustrates the percentage distribution of the scale distortion intervals in the variational projection for the spherical cartographic trapezoid (green column), the intervals in the minimaximal projection for the spherical cartographic trapezoid (orange column), and the map projection using the PE solved by the Ritz method of the territory of the Slovak Republic, bounded by a cartographic trapezoid (yellow column). It can be seen from the above graph that the variational projection in the oblique aspect, which minimizes the Airy-Kavraiskii criterion, optimizes the distribution of the distortions on the projected area. A comparison of the projections in the oblique aspect on the graph shows that the scale distortion in the variational projection, which is in the interval from -2 cm/km to 2 cm/km, is located in 43.3 % of the territory, and in the cartographic projection using the PE solved by the Ritz method, only in 26.5 %. The minimaximal projection in the oblique aspect, shown in the orange bar, is located for the interval from -

2 cm/km to 2 cm/km in 36.46 % of the given territory. Also, the scale distortion in the variational projection with values in the interval from -4 cm/km to 4 cm/km, is found in almost the whole area, namely in 97.7 % of the territory, and in the cartographic projection using the PE solved by the Ritz method, only 52.4 % of the territory. For the minimaximal projection bounded by a spherical cartographic trapezoid, scale distortion values from the interval -4 cm/km to 4 cm/km are located in 73.42 % of the discretization area. The values show that there is less distortion in a larger area in the variational projection (2.6 cm/km), which is more effective than in the cartographic projection using the PE, which is created by the Ritz method (3.4 cm/km), and the minimximal projection (3.2 cm/km).

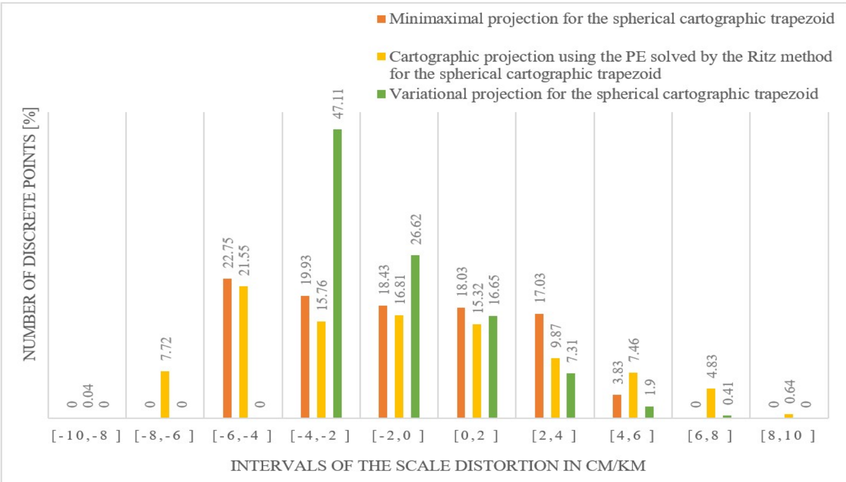


**Fig. 4.8** The percentage distribution of the scale distortion values in the territory of Slovakia according to the intervals in the mentioned projection in the oblique aspect

**Tab. 4.1** The characteristic values of the scale distortions in the analyzed cartographic projections in an oblique aspect

| Method of calculation | Scale distortion [cm/km] | | Airy – Kavraiskii sum criterion [cm/km] |
|---|---|---|---|
| | From | To | |
| **Variational projection for the spherical cartographic trapezoid** | -4.0 | 7.1 | 2.6 |
| **Minimaximal projection for the spherical cartographic trapezoid** | -5.9 | 5.9 | 3.2 |
| **Cartographic projection using the PE solved by the Ritz method for the spherical cartographic trapezoid** | -6.8 | 8.6 | 4.1 |

### 4.2 Modification of the cartographic projections using the Laplace equation solved by the Finite Difference Method and the Finite Elements Method in the territory of the Slovak Republic

The modification of the cartographic projections based on solving the Laplace equation by the Finite Element Method was applied to four proposed cartographic projections, namely, Lambert's conformal conic projection in a polar aspect, Conformal conic projection with minimizing Root Mean Square of the scale distortion values in a polar aspect, Conformal conic projection in an oblique aspect, and Conformal conic projection with minimizing RMS of the scale distortion values in an oblique aspect. Next, we will focus on the proposals of the cartographic projections in the oblique aspect.

At the beginning of the numerical solution, we obtained 211 discretization points with ellipsoidal coordinates $\varphi$, $\lambda$ on the boundary of the domain $\partial\Omega$. The ellipsoidal coordinates $\varphi$ and $\lambda$ were transformed into the Cartesian coordinates $x$ and $y$. At the boundary, we prescribed the scale distortion factor $m$. The numerical solution was calculated in ANSYS 2019 R3 (ANSYS, 2019).

In Tab. 4.2 there are listed the scale distortion values calculated in the proposal of the **Conformal conic projection in an oblique aspect**, the **Conformal conic projection with minimizing RMS of the scale distortion values in an oblique aspect**, and modifications of both projections using LE solved by numerical method FEM. The numerical solution contains all values of the scale distortion factor $m$ for all 52, 015 discrete points in the case of the Conformal conic projection in an oblique aspect, and for all 65, 870 discrete points in the Conformal conic projection with minimizing RMS of the scale distortion values in an oblique aspect, which is the results of the numerical method FEM with Cartesian coordinates $x$ and $y$ of the domain $\Omega$.

In terms of the extreme criteria, the proposal of **Conformal conic projection in an oblique aspect**, which is designed using the criterion of the same absolute value of the scale distortion at the extreme and standard parallels, when the maximum scale distortion is ±5.4 cm/km, it has the best results. The maximum values of the scale distortion in the modification of mentioned projection using LE solved by FEM are almost the same, there are from -5.4 cm/km to +5.8 cm/km. The scale distortion values in the Conformal conic projection in an oblique aspect are in intervals from -3.6 cm/km to 7.6 cm/km. The values of the scale distortion are similar in the modified projection using LE solved by FEM, whose maximum values are from -4 cm/km to 8 cm/km.

According to the Airy-Kavraiskii variational criterion applied to the territory of Slovakia, it has the best values for the modification of the **Conformal conic projection with minimizing RMS in an oblique aspect using LE solved by FEM**, namely 1.9 cm/km. The Conformal conic projection in an oblique aspect has according to the Airy-Kavraiskii variational criterion 4.0 cm/km. In comparison, this criterion is achieved in the modification of the mentioned projection 2.1 cm/km, which is twice as small as in the original proposal. According to Tab. 4.2, it can be said that the values obtained in the modified proposals are the smallest, so they have a significantly more efficient distribution of scale distortions throughout the territory.

In the following figures (Fig. 8 – 11), the $x$ – coordinates are approximately in the interval from 76,000 m to 505,000 m, and the $y$ – coordinates are approximately in the interval from 24,000 m to 218,000 m. Fig. 4.8 presents the isometric lines of the scale distortion factor $m$ for the territory of the Slovak Republic in the proposal of the Conformal conic projection in an oblique aspect. The isometric lines of the scale distortion factor $m$ in the modification of the Conformal conic projection in an oblique aspect using the Laplace equation solved by FEM are shown in Fig. 4.9. In Fig. 10, the isometric lines of scale distortion factor $m$ in the Conformal conic projection with minimizing RMS of the scale distortions in an oblique aspect are presented. Fig. 4.11 shows the isometric lines of the scale distortion factor $m$ for the territory of Slovakia for the modification of the mentioned projection using LE solved by the numerical method FEM. The undistorted isometric lines are marked in blue in all figures.

**Tab. 4.2** The scale distortion values of the domain Ω in the conformal conic projections in an oblique aspect and their modifications using LE solved by FEM

| Method of calculation | Scale distortion values [cm/km] | | The Airy-Kavraiskii variational criterion |
|---|---|---|---|
| | From | To | |
| **Conformal conic projection in an oblique aspect** | -5.4 | 5.4 | 4.0 cm/km |
| **Modification of Conformal conic projection in an oblique aspect using LE solved by FEM** | -5.4 | 5.8 | 2.1 cm/km |
| **Conformal conic projection with minimizing RMS in an oblique aspect** | -3.6 | 7.6 | 2.7 cm/km |
| **Modification of Conformal conic projection with minimizing RMS in an oblique aspect using LE solved by FEM** | -4.0 | 8.0 | 1.9 cm/km |

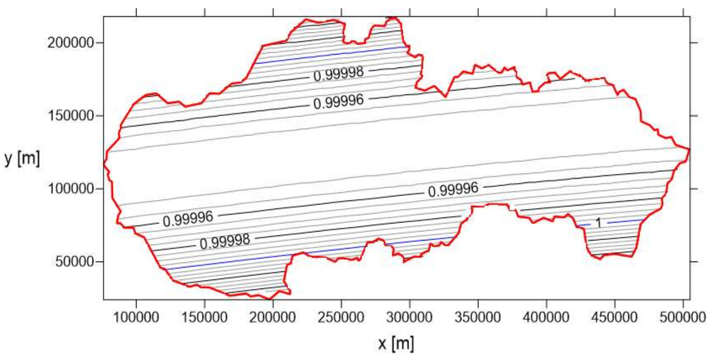


**Fig. 4.8** Isometric lines of the scale distortion factor $m$ for the Conformal conic projection in an oblique aspect

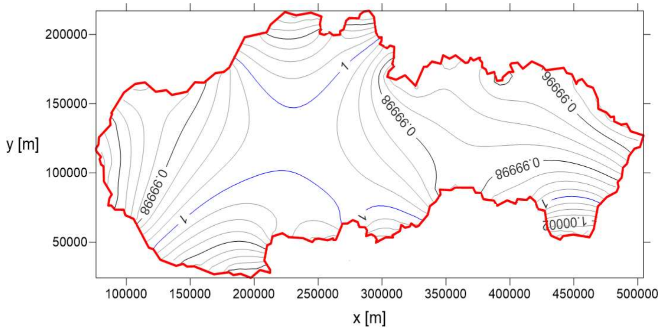

**Fig. 4.9** Isometric lines of the scale distortion factor *m* for modification of the Conformal conic projection in an oblique aspect using LE solved by FEM

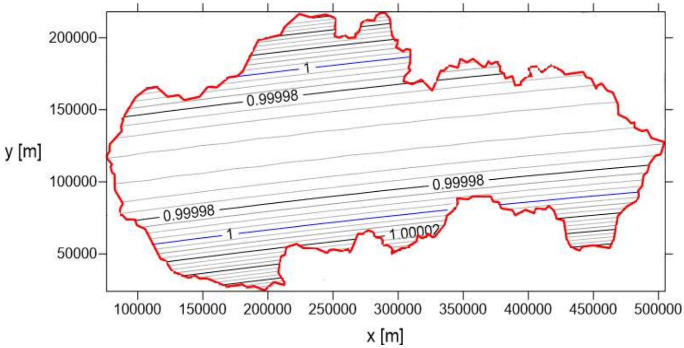

**Fig. 4.10** Isometric lines of the scale distortion factor m for the Conformal conic projection with minimizing RMS in an oblique aspect

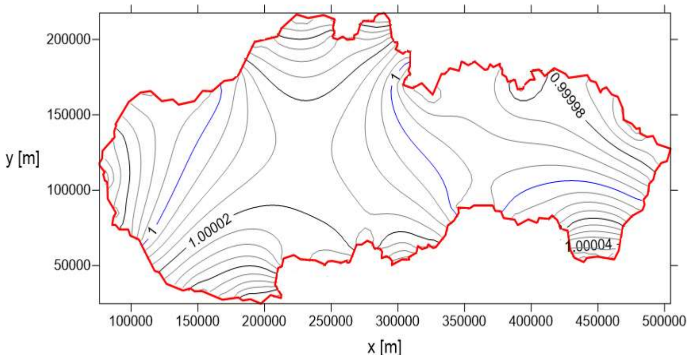

**Fig. 4.11** Isometric lines of the scale distortion factor m for modification of the Conformal conic projection with minimizing RMS in an oblique aspect using LE solved by FEM

The percentage representation of the values of the scale distortion on the territory of the Slovak Republic according to the given intervals (range 2 cm/km) in the conformal conic projections in an oblique aspect and their modifications using LE solved by FEM is shown in graph Fig. 4.12.

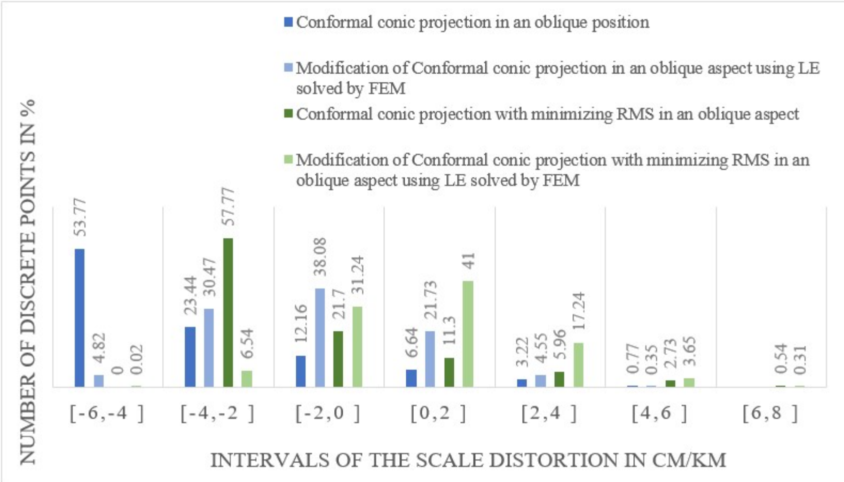


**Fig. 4.12** The percentage representation of the scale distortion values on the territory of Slovakia according to the intervals in the conformal conic projections in an oblique aspect and their modifications using LE solved by FEM

The scale distortions in the Conformal conic projection in an oblique aspect are indicated by a dark blue column, which shows that almost 45 % of the area has a scale distortion in the range from -4 cm/km to 4 cm/km. The Conformal conic projection in an oblique aspect with minimizing RMS, which is shown in dark green, shows that up to 96 % of the projected territory has scale distortion values from the equal interval, which is twice as much as in the previous projection. We can see, that the modified Conformal conic projection with minimizing RMS in an oblique aspect has smaller distortion over a larger area. In graph Fig. 4.12, the modifications of the mentioned projections using the Laplace equation solved by FEM are expressed by light blue, resp. light green color. Based on these results we can see that projections by the numerical method for solving LE by FEM have a better distribution of the scale distortions throughout the territory.

**Conclusions**

The presented thesis on the optimization of the cartographic projections was conducted to determine the general approaches for obtaining the projections using the Airy-Kavraiskii criterion, especially to derive the coefficients for various cartographic projections for the territory of the Slovak Republic. The application of the alternative mathematical methods in the cartographic projection proved to be an interesting factor in our analysis as was the optimization of distortion values, as well as their distribution on the area of the projected territory. Since the transformation will generally change the original proportions, it is important to adopt the distortions of the scales as the basic parameter for the evaluation of the cartographic projections. As the qualitative measure of the map projections, we decided to use the Airy – Kavraiskii criterion (1.13).

The dissertation thesis presents methods for solving the Dirichlet problem in mathematical cartography, dividing them into two different groups. The first group presents solutions, in which the result is an approximate analytical function, i.e. the approximation of a harmonic function that perfectly satisfies the boundary conditions. One of the best known of these methods is the Ritz method based on the solution of Partial Differential Equations, especially on the solution of the Poisson equation. The second group of methods provides a rigorous harmonic function that does not perfectly satisfy the boundary condition. The most suitable method from the second group is the Method of Least Squares, which leads to the solution of the variational, resp. minimaximal conformal cartographic projections. The boundary of the mapping domain is usually approximated by a series of discrete points, so the boundary will be a closed polygon and the isometric line of the scale distortion will be a smooth curve that approximates the polygon.

The variational projection, the minimaximal projection, and the cartographic projection using the Poisson equation solved by the Ritz method were compared. In the variational and minimaximal cartographic projections, we proceed from the evaluation of the projections according to some variational, resp. the minimaximal criteria described in section 1.5. The basic idea of the variational, resp. minimaximal projections is to find such a function $\ln(m)$, which minimizes the value of the Airy-Kavraiskii criterion for conformal projections (1.13), which satisfies the solution of the Laplace equation. The cartographic projection, which uses the Poisson differential equation (1.9) with the right-hand side $f$, is solved using the Ritz method. These projections were created for the territory defined in various ways, specifically for the spherical geographic trapezoid, the spherical cartographic trapezoid, and the area using the real border of the country.

In terms of the normal and oblique aspects, a comparison of the analyzed cartographic projections in the same aspects was performed. In the analyzed cartographic projections in a normal and an oblique aspect, we calculated the characteristics of the scale distortions. All projections are conformal, so they preserve the angles. The above analysis showed that the extreme values of the scale distortions are much smaller than the criterion ±10 cm/km required by the Geodesy, Cartography and Cadastre Authority of the Slovak Republic, except for the variational projection for a spherical geographic trapezoid. Our analysis showed that the variational projection of Slovakia achieves more satisfactory scale distortion values than the cartographic projection using the Poisson equation solved by the Ritz method. The advantage of the Ritz method is that it is possible to choose a boundary condition for a predefined undistorted convex closed curve.

The dissertation thesis also presents various types of conical projections suitable for the territory of the Slovak Republic, in the normal and oblique position (Vajsáblová, 2015) and their modifications using the Laplace equation solved by the numerical method, namely the Finite Element Method (FEM). The parameters of these conical projections are determined from the criteria for distortion of parallel circles in the projected area, as well as optimization of the scale distortion by the method of minimizing the Root Mean Square value of the scale distortion in the projected area using the Airy-Kavraiskii criterion. The scale distortion values in the cartographic projections are modified using the Laplace equation solved by FEM with boundary conditions on the basis of Cartesian coordinates $x$, $y$, and values of the scale distortions in the Slovak Republic given by its boundary.

The performed analysis includes their mutual comparison, as well as a comparison with modified projections, which were created from the solution of the Laplace equation using the numerical method FEM. The presented analysis showed a positive effect of using the Laplace equation solved by the Finite Element Method on the scale distortion values in the proposed cartographic projections. From the comparison of the mentioned projections and their modifications using LE solved by FEM, based on the achieved extreme-scale distortions and the optimal distribution of the scale distortions in the area of the Slovak Republic, we conclude the following conclusions:

- From the point of view of the normal and oblique aspect of the cartographic projections, the normal aspect, from which a smaller number of transformation steps results is more advantageous. It is a direct projection of the reference ellipsoid GRS80 on the projection plane, which also simplifies the reverse transformation steps.

- In terms of the extreme scale distortions, the most effective scale distortions (± 5.4 cm/km) have the Conformal conic projection in the oblique aspect proposed in (Vajsáblová, 2015).

- From the comparison of the scale distortions, we can conclude that all proposed cartographic projections, as well as their modifications using the Laplace equation solved by FEM, achieve lower values of scale distortions, and also the values of the Airy-Kavraiskii criterion than the currently used Křovák's conformal conic projection (from -10 cm/km to 11 cm/km). The proposed projections and their modifications using the Laplace equation have a significantly more efficient distribution of scale distortions throughout the territory.

- From the point of view of the statistical distribution of the scale distortion on the area, the most advantageous projection shows the Conformal conic projection with minimizing RMS of the scale distortion values in an oblique aspect, while the isometric lines of the scale distortion are in the image of cartographic parallel circles.

- Modified projections using the Laplace equation solved by FEM have a minimized value of the scale distortions in the area using the Airy-Kavraiskii variational criterion (3.2 and 2.4 cm/km, respectively 2.1 and 1.9 cm/km).

The dissertation was solved based on the theses specified in its assignment. From the theoretical benefits, we would like to highlight the derivation of specific members of the map equations for the cartographic projection using the Poisson equation solved by the Ritz method. Furthermore, we also consider the formulation of the methodology for cartographic projections using the PE solved by the Ritz method to be beneficial, especially for the 4$^{th}$ characteristically equation from the family of functions $u$ (3.42), which contains more coefficients, and clarify the results of the scale distortion values.

We also consider the original application of the solution of the Laplace equation using FEM in the field of cartographic projections to be a contribution to the field of applied mathematics. In all applications, we have also shown the advantages and benefits of the proposed cartographic projections compared to the currently used Křovák's conformal conic projection.

The presented thesis opens up further possibilities for solving the problem of cartographic projections using Partial Differential Equations, especially the Poisson equation and the Laplace equation. The advantage of the Ritz method is that it is possible to choose a boundary condition for a predefined undistorted convex closed curve. However, such a defined system with a larger number of points is very difficult to solve. In addition to the computational disadvantage, the polygon must be strictly convex. For this, convex, non-symmetric polygon function $u$, which leads to the solution, contains as many polynomial members as there are points in the convex boundary of the boundary, which leads to computational disadvantages.

## References


ÁBRAHÁMOVÁ, A. – VAJSÁBLOVÁ, M. (2022): A comparison of variational projection and cartographic projection by Ritz's method. In *Slovak Journal of Civil Engineering*, *2022* - No. SJCE-D-22-00002R2 – ACCEPTED 6.5.2022

ÁBRAHÁMOVÁ, A. – VAJSÁBLOVÁ, M. – MACÁK, M. (2022): Modification of distortions in the conical cartographic projection of Slovakia using Laplace equation solved by the Finite Element Method. In *Kartografické listy/Cartographic letters*, 2022, 30 (1), xxx – ACCEPTED 11.5.2022

ÁBRAHÁMOVÁ, A. (2021): Ritz method in the mathematical cartography. In *Advances in Architectural, Civil and Environmental Engineering*. Bratislava: Spektrum STU, 2021, ISBN 978-80-227-5150-6, pp. 13-18.

ÁBRAHÁMOVÁ, A. (2020): Application of the finite element method in the mathematical cartography. In *Advances in Architectural, Civil and Environmental Engineering*. Bratislava: Spektrum STU, 2020. ISBN 978-80-227-5052-3, pp. 14-21.

ANSYS ®. Academic Research Mechanical, Release 18.1.

ČEBYŠEVSKIJ, P. L. (1962): *Konštrukcia geografickej mapy*. Výber z diela. Chelsea, 1962.

FRANKICH, K. (1982): *Optimization of geographic map projections for Canadian territory*: dissertation thesis. Canada, Simon Fraser University, 1982. 241 pp.

GANDER, M. J. – WANNER, G. (2012): From Euler, Ritz, and Galerkin to Modern Computing. In SIAM REVIEW, *Society for Industrial and Applied Mathematic*s. ISSN 1095-7200, 2012, Vol. 54, No. 4, pp. 627–666.

GOETZ, A. (1970): *Introduction to Differential Geometry.* New York: Addison Wesley Publishing Company, 1970.

GRAFAREND, E. W. – KRUMM, F. W. (2006): *Map Projections. Cartographic Information Systems*. Springer-Verlag Berlin Heidelberg 2006, ISBN 978-3-642-07178-2.

HOFFMANN, Z. – HOJOVEC, V. (1976): Konformní zobrazení asymetrické oblasti podle Čebyševova kritéria. In *Geodetický a kartografický obzor*. ISSN 0016-7096, 1976, roč. 22 (64), č. 6, s. 151-153.

HOJOVEC, V (1996): Aplikace optimalizačních kritérií v konformních zobrazeních. In *Geodetický a kartografický obzor*. ISSN 0016-7096, 1996, roč. 42 (84), č. 7, s. 133-138.

HOJOVEC, V. – BOŘÍK, M. – MIKUTA, V. – MINÁŘ, P. (1997): Výsledky optimalizace konformního kartografického zobrazení pro Českou republiku. In *Geodetický a kartografický obzor*. ISSN 0016-7096, 1997, roč. 43 (85), č. 12, s. 253-256.

KAVRAJSKIJ, V. V. (1959): *Collected works 2*. Moscow, 1959.

KESSLER, F.C., BATTERSBY, S.E. (2019): *Working with map projections – A Guide to Their Selection.* Padstow: CRC Press Taylor & Francis Group, 2019. pp. 300. ISBN 978-1-138-30498-7.

MESHCHERYAKOV, G. A. (1968): *Teoreticheskie osnovi matematicheskoy kartografii*. Nedra, Moscow, 1968.

REDDY, J. N. (1993): *An introduction to the finite element method.* New York: McGraw-Hill, 1993.

SCHMID, E. (1974): *World Maps on the August Epicycloidal Conformal Projection*. Technical Report NOS 63, Rockville: NOAA, 1974. 22 p.

SKOŘEPA, Z. – DUŠEK, R. (1997): Řešení konformního zobrazení České republiky pomocí programu MATLAB. In *Geodetický a kartografický obzor.* ISSN 0016-7096, 1997, roč. 43 (85), č. 6, s. 119-124.

SZATMÁRI, D. (2014): Optimization of conformal cartographic projections. In *Advances in Architectural, Civil and Environmental Engineering*. Bratislava: Slovenská technická univerzita v Bratislave, 2014. ISBN 978-80-227-4301-3, pp. 54-60.

SZATMÁRI, D. (2015): Optimization of conformal cartographic projections for the Slovak Republic according to Chebyshev's theorem. In *Slovak Journal of Civil Engineering*. ISSN 1338-3973, 2015, 23 (4), pp. 19-24

SZATMÁRI, D. – VAJSÁBLOVÁ, M. – MOJŠOVÁ, D. (2017): Application of Evaluation Criteria to Cartographic Projections. In *Acta Polytechnica Hungarica*, ISSN 1785-8860, 2017, Vol. 14, Issue Number 5, pp. 11-27.

URMAJEV, N. A. (1947): *Methods for finding new cartographic projections*. Moscow: Trudy CNIIGAiK, 1947.

VAJSÁBLOVÁ, M. (2011): Návrh nového kartografického zobrazenia územia Slovenskej republiky. In *Geodetický a kartografický obzor*. ISSN 0016-7096, 2011, roč. 57 (99), č. 8, s. 185-190.

VAJSÁBLOVÁ, M. (2015): *Aspekty tvorby kartografického zobrazenia pre územie Slovenska.*1. vyd. Bratislava: Slovenská technická univerzita v Bratislave. Edícia vedeckých prác. ISBN 978-80-227-4393-8, 108 s.

VAJSÁBLOVÁ, M. (2021): *Matematické základy kartografie* 1. vyd. Bratislava: Spektrum STU, Edícia vysokoškolských učebníc, 2021. 443 s. ISBN 978-80-227-5114-8.

ZIENKIEWICZ, O. C. (1977): *The Finite Element Method* (*3rd ed*). McGraw-Hill: New York, 1977.

**List of published works of the author**

**AFD Publikované príspevky na domácich vedeckých konferenciách**

AFD01 ÁBRAHÁMOVÁ, Andrea. One of the approaches to the creation of conformal projections. In *Advances in Architectural, Civil and Environmental Engineering [elektronický zdroj]: 29th Annual PhD Student Conference on Applied Mathematics, Applied Mechanics, Building Technology, Geodesy and Cartography, Landscaping, Theory and Environmental Technology of Buildings, Theory and Structures of Buildings, Theory and Structures of Civil Engineering Works, Water Resources Engineering. October 16th 2019, Bratislava*. 1. vyd. Bratislava : Spektrum STU, 2019, CD-ROM, s. 13-19. ISBN 978-80-227-4972-5.

AFD02 ÁBRAHÁMOVÁ, Andrea - VAJSÁBLOVÁ, Margita. Kinetic Curves in Cartographic Projections. In *Proceedings of the slovak-czech conference on geometry and graphics 2019 [elektronický zdroj /]*. 1. vyd. Bratislava : Vydavateľstvo SCHK, 2019, S. 27-40. ISBN 978-80-8208-024-0.

AFD03 ÁBRAHÁMOVÁ, Andrea. Application of the finite element method in the mathematical cartography. In *Advances in Architectural, Civil and Environmental Engineering [elektronický zdroj]: 30th Annual PhD Student Conference on Applied Mathematics, Applied Mechanics, Building Technology, Geodesy and Cartography, Landscaping, Theory and Environmental Technology of Buildings, Theory and Structures of Buildings, Theory and Structures of Civil Engineering Works, Water Resources Engineering. October 14th 2020, Bratislava, Slovakia*. 1. vyd. Bratislava : Spektrum STU, 2020, CD-ROM, s. 14-21. ISBN 978-80-227-5052-3.

AFD04 ÁBRAHÁMOVÁ, Andrea. Ritz method in the mathematical cartography. In *Advances in Architectural, Civil and Environmental Engineering [elektronický zdroj]: 31st Annual PhD Student Conference on Applied Mathematics, Building Technology, Geodesy and Cartography, Landscaping, Theory and Environmental Technology of Buildings, Theory and Structures of Buildings, Theory and Structures of Civil Engineering Works, Water Resources Engineering. October 13th 2021, Bratislava, Slovakia*. 1. vyd. Bratislava : Spektrum STU, 2021, CD-ROM, s. 13-18. ISBN 978-80-227-5150-6.

**AFG Abstrakty príspevkov zo zahraničných konferencií**

AFG01 ÁBRAHÁMOVÁ, Andrea. Numerical methods in mathematical cartography. In *ISCAMI 2020: proceedings of the 21th International Student Conference on Applied Mathematics and Informatics. Malenovice, Czech Republic, September 8-11, 2020*. 1. vyd. Ostrava : University of Ostrava, 2020, s. 21-22. ISBN 978-80-7599-199-7.

**In progress (accepted):**

ADN01 ÁBRAHÁMOVÁ, Andrea - VAJSÁBLOVÁ, Margita: A comparison of variational projection and cartographic projection by Ritz's method. In Slovak Journal of Civil Engineering, 2022 - No. SJCE-D-22-00002R2 – ACCEPTED 6.5.2022

ADN02 ÁBRAHÁMOVÁ, Andrea - VAJSÁBLOVÁ, Margita – MACÁK, Marek: Modification of distortions in the conical cartographic projection of Slovakia using Laplace equation solved by the Finite Element Method. In Kartografické listy/Cartographic letters, 2022, 30 (1), xxx – ACCEPTED 11.5.2022