SLOVAK UNIVERSITY OF TECHNOLOGY IN BRATISLAVA
FACULTY OF CIVIL ENGINEERING

Ing. Ladislav Sátor

Report of dissertation thesis

MESHLESS FORMULATIONS FOR SOLUTION OF BOUNDARY VALUE PROBLEMS IN CONTINUUM MECHANICS

to obtain the academic degree *philosophiae doctor*, PhD.

in doctoral study programme

5.1.7 Applied Mechanics

Bratislava **2015**

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The dissertation thesis was elaborated during the full-time form of doctoral study programme at the Institute of Construction and Architecture, Slovak Academy of Sciences, which is an external educational institution for Faculty of Civil Engineering SUT in Bratislava.

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Report was sent on ....

Defense of the dissertation thesis take place on:

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Dean of the Faculty of Civil Engineering SUT

Abstract

This dissertation thesis aims to contribute to further understanding of phenomena by development of advanced meshless formulations for solution of general problems of continuum mechanics. Meshless computational techniques for numerical analysis of bending problems of beams and plates are proposed. There is presented the derivation of the unified formulation of governing equations for homogeneous and functionally graded (FG) plates, which includes the Kirchhoff-Love theory (KLT) as well as the 1st and 3rd order shear deformation plate theory (SDPT). Moreover, the formulation is extended to describe the physical processes of coupled thermoelasticity in bending problems of functionally graded plates. To facilitate the numerical solution of rather complex governing equations, we propose the strong formulation combined with Moving Least Square (MLS) approximation technique and/or Point Interpolation Method (PIM) for field variables and their derivatives with preserving the physical nature of problems.

The numerical experiments presented in this dissertation thesis deal with static and dynamic analysis of circular and/or square plates. Homogeneous as well as functionally graded material (FGM) properties are considered for both the elastic and coupled thermoelastic problems. The weak point of conventional strong meshless formulations for problems involving high order derivatives is low accuracy and computational inefficiency due to such derivatives. This problem has been successfully overcome by decomposing the original problem into coupled problems with lower order derivatives.

Results presented in the thesis show that, the proposed and developed meshless formulations exhibit excellent rate of convergence of accuracy, and acceptable computational efficiency. Furthermore, there is revealed lot of new phenomena and coupling effects due to multifield interactions appearing in FGM plates (beams).

Keywords

functionally graded materials; Kirchhoff-Love thin plate bending theory; 1st and 3rd order shear deformation plate theory; static, dynamic and thermoelastic problems; coupling effects; meshless approximation; strong formulation; decomposition technique

Abstrakt

Táto dizertačná práca si kladie za cieľ prispieť k hlbšiemu pochopeniu javov vývojom progresívnych bezprvkových formulácií pre riešenie všeobecných okrajových úloh mechaniky kontinua. Pre účely numerickej analýzy okrajových úloh ohybu nosníkov a dosiek je rozpracovaná bezprvková výpočtová metóda. V práci je prezentované odvodenie zjednotenej formulácie riadiacich rovníc pre okrajové úlohy dosiek s homogénnymi alebo funkcionálne gradovanými (FG) materiálovými vlastnosťami, ktoré v sebe zahŕňa klasickú teóriu ohybu tenkých dosiek (Kirchhoff-Love teória - KLT) ako aj šmykovo-deformačnú teóriu ohybu dosiek prvého a tretieho rádu (SDPT). Táto formulácia je rozšírená aj na popísanie fyzikálnych javov a procesov pri ohybe funkcionálne gradovaných dosiek v termoelasticite. Z dôvodov uľahčenia numerického riešenia pomerne zložitých riadiacich rovníc, navrhujeme silnú formuláciu v kombinácii s aproximačnou technikou podľa Pohyblivej metódy najmenších štvorcov (MLS) alebo Bodovej interpolačnej metódy (PIM).

Numerické experimenty uvedené v tejto dizertačnej práci sú venované statickej a dynamickej analýze kruhových a štvorcových dosiek. V numerických simuláciach sú uvažované elastické a viazané termoelastické okrajové úlohy pre dosky s homogénnymi, ako aj funkcionálne gradovanými materiálovými (FGM) vlastnosťami.

Slabou stránkou konvenčnej silnej bezprvkovej formulácie pre úlohy s deriváciami vysokých rádov je nízka presnosť a výpočtová efektívnosť z dôvodu týchto derivácii poľných premenných. Tento nedostatok bol úspešne odstránený rozkladom pôvodnej okrajovej úlohy na systém viazaných okrajových úloh s nižšími deriváciami poľných premenných.

Z výsledkov prezentovaných v tejto práci vyplýva, že navrhované a rozpracované bezprvkové formulácie vykazujú vynikajúcu konvergenciu presnosti a prijateľnú výpočtovú efektívnosť. Naviac v práci je odhalených vaicero javov a efektov previazanosti vznikajúcich v dôsledku interakcií polí vo funkcionálne gradovaných doskách a nosníkoch.

Kľúčové slová

funkionálne gradované materiály; klasická teória dosiek; teória šmykových deformácii dosiek prvého a tretieho rádu; efekty previazanosti; dekompozícia; bezprvkové aproximácie; termoelasticita

Contents

1 Introduction

Continuum mechanics is a branch of physical sciences that deals with the deformation and motions of continuous material media under the influence of external effects. Mathematical models of the continuum mechanics are usually formulated as initial-boundary value problems for partial differential equations being served as governing equations. In most cases, these equations are difficult to be solved analytically in a closed form, and we have to find approximation methods to the numerical solution of boundary value problems. The mathematical foundations of most popular approximation methods of solutions of the boundary value problems (BVP) in engineering resp. continuum mechanics were first time published by Galerkin [1] and Ritz [2].

In the 20th century with the development of high performance computers, simultaneously got the computer-based modelling and simulation of structures more and more important role in the engineering computations. During the century several approximation methods were developed for the solutions of BVP in engineering practice, for example the Finite Element Method (FEM)([3], [4]), Finite Volume Method (FVM) [5], Boundary Element Method (BEM) [6], Meshless methods ([7], [8]), etc.

Plates are considered as the one of the most important components of engineering applications. They are widely used in civil engineering, aerospace crafts and other branches of advanced engineering practice. Several plate theories have been developed for analysis of thin and/or thick elastic plates. It is well known that in the Kirchhoff-Love theory (KLT) [9] of bending of thin elastic plates the shear deformations are omitted and the deflections can be calculated separately from the in-plane deformations in homogeneous plates. Besides of KLT there have been developed generalized shear deformation theories including the first-order shear deformation theory (FSDT) [10] and higher-order shear deformation theory (HSDT) [10] which account for transverse shear strains and stresses in contrast to KLT.

In the last decades of previous century the application of laminated composite structures became to the part of engineering practice.

Besides the advantages of such structures, the main disadvantage consists in delamination due to discontinuities of stresses on interfaces. A great effort has been done to overcome this shortcoming.

Nowadays the modern technologies of material engineering such as functionally graded materials (FGM) with continuous spatial variation of properties enable us to replace the laminated composites by FGM [11].

The formulation of boundary value problems for structures with such a complex material properties leads to complicated governing equations described by partial differential equations with variable coefficients. One of the main objectives of this thesis is the correct formulation of such boundary value problems and their accurate and stable solution by meshless methods.

1.1 Propositions of the dissertation thesis

The aims of the thesis are:

- Proposal and development of new meshless formulations for numerical solution of plate bending problems, with implementing in computer codes
- Proposal and development of meshless formulations for coupled problems of continuum mechanics (e.g. thermal loading of plates) with implementing in computer codes.
- Development of reliable numerical techniques for the solution of plate problems with including continuously variable material coefficients and/or other design parameters.
- Investigation of the numerical stability, convergence and efficiency of proposed computational schemes.

2 Current trends in solutions of boundary value problems in continuum mechanics

The effects and processes in continuous media can be described faithfully within the theory of continuous media which is a phenomenological theory dealing with macroscopic characteristics of the media instead of a microscopic description. The macroscopic state in a continuous medium is characterized by classical fields obeying certain governing equations (or equations of motion) and constitutive relationships with the current state being determined by external impacts and specified initial-boundary conditions. Thus, a typical task in continuum mechanics is to solve an initial-boundary value problem for some partial differential equations (PDE). The closed form solutions by using analytical methods are available only for simplified problems, while in engineering practise we need to use numerical method to get approximate solutions.

To overcome the mathematical difficulties of solution of initial-boundary value problems in real engineering practise, various discretization methods have been proposed and developed. The field variables are replaced by certain approximations within subdomains with reducing the infinite number of degrees of freedom into a finite one represented by nodal unknowns. The nodal unknowns are calculated from a system of algebraic equations resulting either from the collocation of the governing equations and the prescribed boundary conditions at certain collocation points (strong formulations) or by integrations over finite sub-domains in weak formulations. There have been developed lot of discretization techniques, like the finite strip method ([12],), the finite difference methods (FDM) ([13],), the finite element method (FEM) ([14], [15], [16]),the extended finite element method (XFEM)([17], [18]), the scaled boundary finite element method (SBFEM) ([19],) and the boundary element method (BEM) ([20], [21]). Although these classical discretization methods are well developed and they are still enriched by some new developments (see the above references), in the last two decades a great attention has been paid to the development of mesh-free formulations (see e.g. [22] **)** with using meshless approximations instead of element based approximations such as used in FEM and BEM.

3 Introduction to meshless methods

Meshless methods for solving PDE in engineering and sciences are new powerful alternatives to the element-based computational methods. Focusing on nodes instead of elements used common in the FEM and BEM, the meshless approaches have certain advantages [7].

In the last decades the meshless methods have attracted much attention due to their potential in omitting the need for human-labour intensive process of constructing geometric meshes, remeshing in evolving solutions [7]. This is very useful in problems with moving boundaries, like machining process simulations or penetration problems. Furthermore, numerical difficulties which relate to methods based on element use discretization, like locking and distortion of elements, when element becomes infinitely stiff, are eliminated. One could name also other advantages such as elimination of discontinuities occurring on element interfaces in element-based methods, elimination of numerical instabilities due to large distortion of finite elements, convenient applicability to modelling separable media, etc. The main objective of the meshless methods is to get rid off/or at least reduce the difficulty of meshing and remeshing the entire structure, by adding or deleting nodes in entire structure.

The meshless method was first time presented by Gingold and Monaghan ([23], [24]). It was the smooth particle hydrodynamics (SPH) method for modelling astrophysical phenomena. The development of meshless methods increased rapidly, mainly after publishing of Diffuse Element Method by Nayroles et al. [25]. Several so-called meshless method were developed, like Element Free Galerkin method (EFG) [26], Reproducing Kernel Particle Method (RKPM) [27], the Partition of Unity Finite Element Method (PUFEM) [28] and Meshless Galerkin methods using Radial Basis

Functions (RBF) ([29], [30]). It should be mentioned, some review papers was published about the research of the above mentioned methods ([31], [32]).

The principal difference between these so-called meshless methods is only in the techniques, used for interpolating of trial function of unknown field variables. Even though mesh is not required in these methods for the interpolation of the trial and test function, the use of shadow elements ("background meshes") is necessary for the integration in weak formulations considered in global sense. These methods are not truly meshless.

In the recent decades, truly meshless methods were developed for linear and non-linear boundary value problems of continuum mechanics. Historically older is the meshless local boundary integral equation (LBIE) method by Zhu et al. [33], which appeared to be a special class of the more general meshless local Petrov-Galerkin (MLPG) method in [34] and [35]. In truly meshless methods, the domain and/or boundary meshes are not required either for purpose of interpolation of field variables, or for integration of the governing equations in weak form. All relevant integrals can be easily evaluated over regularly shaped local subdomains (spheres for 3-D and circles for 2-D problems, in general). The MLPG method was employed for solution of static ([36], [37]) and dynamic [38] fracture mechanics problems, as well as for solution of plate bending problems ([39];) and for many other engineering applications in various fields of science.

3.1 Approximation techniques

The approximation (trial) function is the fundamental aspect of every numerical method, with discretizing the continuous media. A weight function, which plays an important role in the performance of the methods, is used in all varieties of meshless methods. The compact support of weight function gives a local character to the meshless methods. The meshless methods construct approximations absolutely in terms of nodal points, in contrast of mesh-based methods.

3.1.1 Moving Least Squares method

In this chapter, we explain the concept of the Central Approximation Node (CAN) MLS approximation proposed by Sladek et al. [40] Without going into details [41], the approximation of field variable $u(\mathbf{x})$ around the central approximation node $\mathbf{x}^q$ can be expressed by

$$u(\mathbf{x}) \approx \sum_{a=1}^{N^q} \hat{u}^{\overline{a}} \phi^{(q,a)}(\mathbf{x}) \ , \quad \overline{a} = n(q,a) \tag{3.1}$$

where $\bar{a}$ is the global number of the a-th node from the influence domain of $\mathbf{x}^q$, N^q is the number of nodal points in the influence domain, and $\phi^{(q,a)}(\mathbf{x})$ is the shape function associated with the node $n(q,a)$. The CAN node can be selected as the nearest node to the field point $\mathbf{x}$.

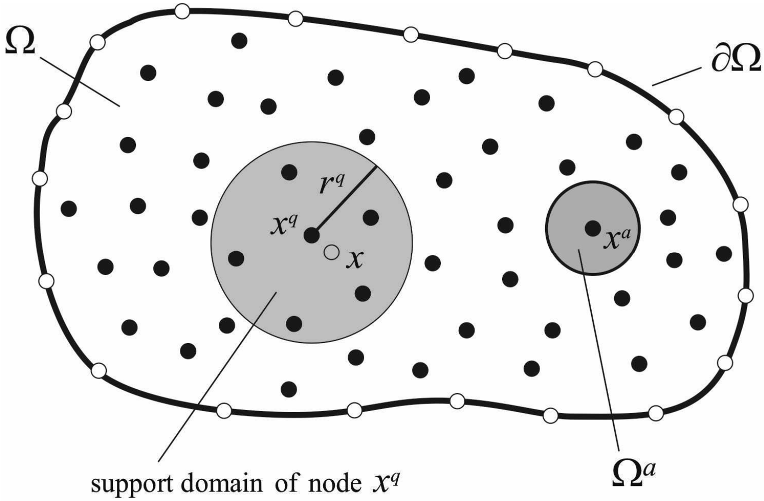

Figure 3.1- Sketch of the support domain of the CAN $\mathbf{x}^q$ for approximation at the point $\mathbf{x}$; local subdomain around the node $\mathbf{x}^a$

The derivatives of the field variable $u(\mathbf{x})$ can be approximated by derivatives of approximated fields (D0-approach) [42], i.e.

$$u_{,ijk...l}(\mathbf{x}) \approx \sum_{a=1}^{N^q} \hat{u}^{\bar{a}} \phi^{(q,a)}_{,ijk...l}(\mathbf{x}) . \qquad (3.2),$$

or by using modified differentiation technique (D1-approach) [41].

3.1.2 Point Interpolation Method

In this chapter, we shall explain the meshless approximation of primary field variables by Point Interpolation Method (PIM). The combination of the radial basis functions and polynomials leads to a sufficiently accurate and numerically stable approximation method. For the RBF, we have used inverse multiquadratics

$$R^n(\mathbf{x}) = \left(\left| \mathbf{x}\text{-}\mathbf{x}^n \right|^2 + (c^n)^2 \right)^{p/2} \tag{3.3}$$

Without going into details [41], the approximation of field variable around the node $\mathbf{x}^q$ can be expressed by

$$f(\mathbf{x}) \approx \sum_{a=1}^{N^q} f^{\overline{a}} \varphi^{(q,a)}(\mathbf{x}) \ , \quad \overline{a} = n(q,a) \tag{3.4}$$

where $\overline{a}$ is the global number of the a-th node from the influence domain of $\mathbf{x}^q$, N^q is the number of nodal points in the influence domain, and $\varphi^{(q,a)}(\mathbf{x})$ is the shape function associated with the node $n(q,a)$. The shape functions satisfy the Kronecker δ property. The derivatives of the field variable can be approximated by derivatives of approximated fields (D0-approach) [42], i.e.

$$f_{,ijk...l}(\mathbf{x}) \approx \sum_{a=1}^{N^q} f^{\overline{a}} \varphi_{,ijk...l}^{(q,a)}(\mathbf{x}) \ . \tag{3.5}$$

or by using modified differentiation technique (D1-approach) [41].

4 Meshless analysis of plate bending problems

Plates are three dimensional structural elements whose thickness is very small as compared with other dimensions. In the plate theories the 3-D formulation of elastic problem is assumed in semi-integral form across the plate thickness, and due to this assumption the original problem is simplified to a 2-D problem. Since the late 19th century several plate theories have been developed for analysis of thin and/or thick plates. It is well known that in the Kirchhoff-Love theory (KLT) of bending of thin elastic plates the shear deformations are omitted and the deflections can be calculated separately from the in-plane deformations in homogeneous plates. Besides of KLT there have been developed generalized shear deformation theories including the first-order shear deformation theory (FSDT)([10],) and higher-order shear deformation theory (HSDT) ([10], [43]) which account for transverse shear strains and stresses in contrast to KLT. Note that in the FSDT the shear stains are represented as constant through the plate thickness and the theory requires shear correction factors to compute transverse shear forces, while in the HSDT the zero tangential traction boundary conditions are satisfied on the surfaces of the plate and no shear correction factor is required.

4.1 Angularly symmetric bending of circular plates

In the Kirchoff-Love theory of bending of thin plates [9] all the physical quantities are expressed in terms of the deflection $w(\mathbf{x})$ and/or its derivatives. For the plate of thickness b and midplane Ω orthogonal to the axis x_3, the tensor of moments can be expressed in terms of the second order derivatives of deflection as

$$M_{ij} = -D\left[(1-\nu)w_{,ij} + \nu\delta_{ij}\nabla^2 w\right], \quad D = \frac{Eb^3}{12(1-\nu^2)} \tag{4.1}$$

where D is the bending stiffness, E and ν is the Young's modulus and Poisson's ratio, respectively.

The governing equation for deflections of thin plane is given as

$$M_{ij,ij}(\mathbf{x}) = -q(\mathbf{x}), \tag{4.2}$$

hence after substituting (4.1$_1$) to (4.2) we can obtain governing equation in the form

$$\left[D(1-\nu)w_{,ij}\right]_{,ij} + \nabla^2\left[\nu D\nabla^2 w\right] = q \tag{4.3}$$

with $q(\mathbf{x})$ being the density of transversal loading applied on the plate surface.

If we shall consider the bending stiffness to be constant, then the governing equation is simplified as

$$D\nabla^2\nabla^2 w = q\ . \tag{4.4}$$

Three basic boundary conditions can be assumed on the boundary edge Γ:

(i) clamped edge: $w\big|_\Gamma = 0\ ;\ \left.\frac{\partial w}{\partial \mathbf{n}}\right|_\Gamma = 0$

(ii) simply supported edge: $w\big|_\Gamma = 0\ ;\ M\big|_\Gamma = 0$ (4.5)

(iii) free edge: $M\big|_{\Gamma} = 0$; $V\big|_{\Gamma} = 0$

The fourth order derivatives of deflections in governing equations can give rise to serious difficulties not only in the strong formulation for numerical solution, but also in the weak formulation owing to inaccurate approximation of high order derivatives of deflections occurring in the integral equations as well as in boundary conditions.

Therefore, it is expedient to introduce the new field variable defined as

$$m(\mathbf{x}) := -D\nabla^2 w(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega \tag{4.6}$$

Then the governing equation (4.4) is split into two equations given by (4.6) and (4.7)

$$\nabla^2 m(\mathbf{x}) = q(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega \tag{4.7}$$

and the boundary conditions can be expressed in terms field variables including new one.

4.1.1 Formulations in polar coordinates

Bearing in mind the symmetry, the problem can be simplified when we use polar coordinates (r, φ) instead of Cartesian coordinates, where

$(x_1, x_2) = (r\cos\varphi,\ r\sin\varphi)$ and $\partial(.)/\partial\varphi \equiv 0$.

Then,

$$w_{,i}(r) = r_{,i}\frac{\partial w(r)}{\partial r}, \qquad w_{,ij}(r) = \frac{1}{r}\left(\delta_{ij} - r_{,i}r_{,j}\right)\frac{\partial w(r)}{\partial r} + r_{,i}r_{,j}\frac{\partial^2 w(r)}{\partial r^2}, \tag{4.8}$$

$$\nabla^2 w(r) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) w(r)$$

Therefore, the governing equations (4.6) and (4.7) can be written now as

$$D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) w(r) + m(r) = 0 \quad \text{or} \quad D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w(r)}{\partial r}\right) + m(r) = 0 \tag{4.9}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) m(r) = -q(r) \quad \text{or} \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial m(r)}{\partial r}\right) = -q(r) \tag{4.10}$$

4.1.2 Numerical examples

In all numerical computations, we have used a uniform distribution of nodal points and the radius of the sub-domain $r_o = 0.1h$ with h being the distance between two neighbour nodes. The other parameters in the MLS-approximation have been taken as: radius of the interpolation domain $r^a = 3.001h$, shape function parameter $c^a = h$, cubic polynomial basis $m = 4$. In the PIM(RBF+P)-approximation, we have chosen: type of RBF – inverse multiquadrics with $p = -1$, number of multiquadrics around each node $N^q = 16$, number of polynomials $M = 7$, shape parameter $c^a = 2h$. As regards the geometry, we shall consider the circular plate without any hole $\Omega = \{\forall(r,\varphi); r \in [0, r_a], \varphi \in [0, 2\pi]\}$.

The accuracy and convergence for numerical solutions of considered b.v.p. is presented in Figure. 4.1. It can be seen that the formulation CPDE(2xPoiss) exhibits convergent and highly accurate numerical solutions.

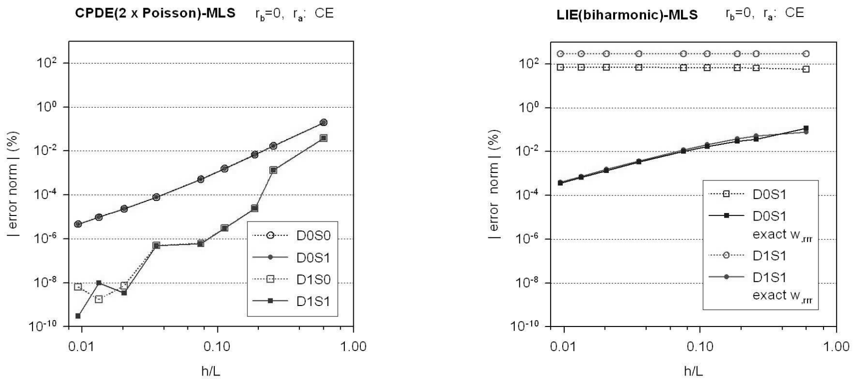

Figure 4.1- Accuracy and convergence of numerical solutions for b.v.p. (A) by CPDE(2xPoiss) and LIE(biharm) combined with MLS-approximations of field variables

From the above study of the accuracy of numerical solutions by **meshless formulations implemented with MLS-approximations of field variables**, the following conclusions can be drawn:

(i) **only the formulations for decomposed problem** (CPDE(2xPoiss)) **yield meaningful results**; the LIE(biharm) formulation fails

(ii) **in the CPDE(2xPoiss), good accuracy and convergence rates are achieved by using the D0-approach**,

(iii) the influence of S1-modification for evaluation of shape functions and their derivatives on accuracy is negligible

The numerical results by CPDE(2xPoiss) shown in Figures. 4.2 for the b.v.p. with clamped edge despite the fact that the accuracy in the formulation by CPDE(2xPoiss) is determined not only by the first order derivatives but also by the second order derivatives. Qualitatively different results have been obtained by the LIE(biharm), where the accuracy of the numerical solution of a boundary value problems is affected also by the accuracy of approximations of the third order derivative $w_{,rrr}(r)$.

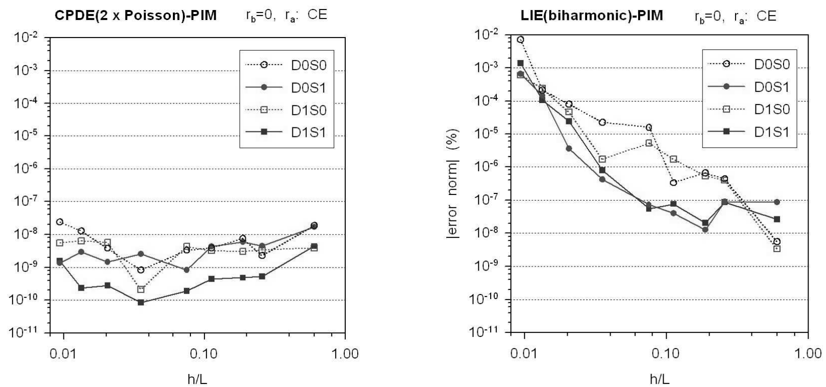

Figure 4.2- Accuracy and convergence of numerical solutions for b.v.p. (A) by CPDE(2xPoiss) and LIE(biharm) combined with PIM-approximations of field variables

Summarizing the study of the accuracy of numerical solution by **meshless formulations implemented with PIM-approximations of field variables**, we conclude:

(i) **the LIE(biharm) formulation gives unreliable results** (convergence is achieved only in the b.v.p. (C))

(ii) **both the LIE(2xPoiss) and CPDE(2xPoiss) formulations give stable and highly accurate numerical solutions** of the b.v.p. (A) and (B); in the case of b.v.p. (C) the accuracy is good with excellent convergence rate

(iii) the influence of S1-modification for evaluation of shape functions and their derivatives on accuracy is negligible

Finally, we pay attention to computational efficiency of various approaches. Since the LIE(biharm) formulation is disqualified owing to unreliable accuracy, and the influence of the advanced evaluation of the derivatives of shape functions (D1-approach) on the accuracy is negligible, we confine to decomposed formulations considered in the weak form (LIE) and strong form (CPDE) with using both the MLS and PIM approximations for field variables and D0S1 evaluation technique. In case of small densities of nodes, the time for creation of the discretized system matrix is a dominant part of the whole time consumption, while in case of large densities of nodes, the time needed for solution of such system of equations is dominant.

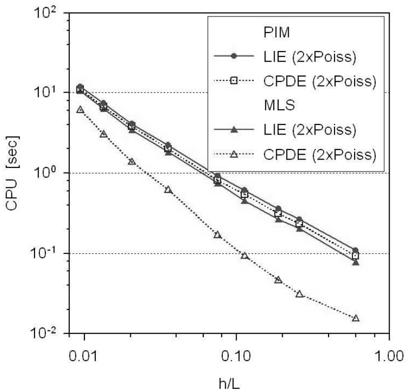

Figure 4.3 - Comparison of computational times spent by two decomposed formulations implemented by both PIM- and MLS-approximations

Furthermore, in the case of LIE(2xPoiss), only the 1st order derivatives are required and there is no principal difference in time consuming for evaluation of these derivatives by the MLS and/or PIM approximation technique. The evaluation of the 2nd order derivatives by the MLS approximation is faster than by the PIM approximation. **Therefore the CPDE(2xPoiss)+MLS is more efficient than the CPDE(2xPoiss)+PIM for low densities of nodes. On the other hand, in case of high densities of nodes, there is practically no difference in the efficiency of the approaches considered in Fig. 4.3.**

4.2 Static analysis of the bending of FGM square plates

In this chapter of we shall investigate the static response of thin and/or thick elastic functionally graded (FG) plates. The spatial variation of material coefficients in the FG composite structures is determined by distribution of volume fractions of particular constituents. The attention is devoted to derivation of the unified formulation of governing equations in FGM plates, which includes the Kirchhoff-Love theory (KLT) as well as the 1st and 3rd order shear deformation plate theory (SDPT).

In order to unify the formulation for all three theories (KLT, FSDPT and TSDPT), we assume displacement field defined by

$$v_i(\mathbf{x}, z) = \delta_{i\alpha}\left\{u_\alpha(\mathbf{x}) + \left[c_1\phi(z) - z\right] w_{,\alpha}(\mathbf{x}) + c_1\phi(z)\varphi_\alpha(\mathbf{x})\right\} + \delta_{i3}w(\mathbf{x}), \qquad (4.11)$$

where $u_\alpha(\mathbf{x})$, $\varphi_\alpha(\mathbf{x})$, $w(\mathbf{x})$ are the in-plane displacements, rotations and deflection fields, respectively, and $\phi(z) := z - c_2\psi(z)$ with $\psi(z) := 4(z)^3 / 3h^2$.

Now, in view of (4.11), the strain tensor is given by

$$e_{\alpha\beta}(\mathbf{x}, z) = \varepsilon_{\alpha\beta}(\mathbf{x}) + \left[c_1\phi(z) - z\right] w_{,\alpha\beta}(\mathbf{x}) + c_1\phi(z)\eta_{\alpha\beta}(\mathbf{x}),$$

$$e_{\alpha 3}(\mathbf{x}, z) = \frac{c_1}{2}\phi'(z)\left[w_{,\alpha}(\mathbf{x}) + \varphi_\alpha(\mathbf{x})\right], \quad e_{33}(\mathbf{x}, z) = 0, \qquad (4.12)$$

where

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right), \quad \eta_{\alpha\beta} = \frac{1}{2}\left(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}\right),$$

in which $\phi'(z) = 1 - c_2\psi'(z)$, $\psi'(z) = 4\left(z / h\right)^2$.

According to the Hooke's law, the 3D elastic stresses in the plate structure are given as

$$\sigma_{ij}(\mathbf{x}, z) = \frac{E}{1+\nu}\left[e_{ij}(\mathbf{x}, z) + \frac{\nu}{H}\delta_{ij}e_{kk}(\mathbf{x}, z)\right]$$

with $H = 1 - \chi\nu$, ($\chi = 2$ for 3D elasticity, while $\chi = 1$ for plane stress formulation) and E, ν being the Young's modulus and the Poisson's ratio, respectively.

In view of strains (4.12), we have

$$e_{kk}(\mathbf{x}, z) = u_{\gamma,\gamma}(\mathbf{x}) - \psi(z)w_{,\gamma\gamma}(\mathbf{x}) + \phi(z)\eta_{\gamma,\gamma}(\mathbf{x}) \qquad (4.13)$$

$$\sigma_{\alpha\beta}(\mathbf{x}, z) = \frac{E}{1-\nu^2}\frac{1-\nu}{H}\left\{\tau^{(u)}_{\alpha\beta}(\mathbf{x}) + c_1\phi(z)\tau^{(\varphi)}_{\alpha\beta}(\mathbf{x}) - \left[c_1\phi(z) - z\right]\tau^{(w)}_{\alpha\beta}(\mathbf{x})\right\}$$

$$\sigma_{\alpha 3}(\mathbf{x},z)=\frac{E}{1+\nu}\frac{c_1\phi'(z)}{2}\left[w_{,\alpha}(\mathbf{x})+\varphi_\alpha(\mathbf{x})\right],$$

$$\sigma_{33}(\mathbf{x},z)=\frac{E\nu}{1-\nu^2}\frac{1-\nu}{H}\left\{u_{\gamma,\gamma}(\mathbf{x})+\left[c_1\phi(z)-z\right](z)w_{,\gamma\gamma}(\mathbf{x})+c_1\phi(z)\eta_{\gamma,\gamma}(\mathbf{x})\right\} \tag{4.13}$$

with

$$\tau^{(u)}_{\alpha\beta}(\mathbf{x}):=H\varepsilon_{\alpha\beta}(\mathbf{x})+\nu\delta_{\alpha\beta}u_{\gamma,\gamma}(\mathbf{x})\,,\quad \tau^{(\varphi)}_{\alpha\beta}(\mathbf{x}):=H\eta_{\alpha\beta}(\mathbf{x})+\nu\delta_{\alpha\beta}\varphi_{\gamma,\gamma}(\mathbf{x})\,,$$

$$\tau^{(w)}_{\alpha\beta}(\mathbf{x}):=Hw_{,\alpha\beta}(\mathbf{x})+\nu\delta_{\alpha\beta}w_{,\gamma\gamma}(\mathbf{x}) \tag{4.15}$$

Having known the transversal dependence, the plate bending problem can be analyzed as a 2D problem. Since $h \ll L$ (where h and L are the thickness and a characteristic width of the plate, respectively), the variations on $z \in [-h/2, h/2]$ can be treated by using the average stresses and stress couples across the thickness of the plate

$$T_{\alpha\beta}(\mathbf{x}):=\int_{-h/2}^{h/2}\sigma_{\alpha\beta}(\mathbf{x},z)dz\,,T_{3\beta}(\mathbf{x}):=\int_{-h/2}^{h/2}\sigma_{3\beta}(\mathbf{x},z)dz$$

$$Q_{3\beta}(\mathbf{x}):=\int_{-h/2}^{h/2}\psi'(z)\sigma_{3\beta}(\mathbf{x},z)dz\,,M_{\alpha\beta}(\mathbf{x}):=\int_{-h/2}^{h/2}z\sigma_{\alpha\beta}(\mathbf{x},z)dz\,,$$

$$S_{\alpha\beta}(\mathbf{x}):=\int_{-h/2}^{h/2}\psi(z)\sigma_{\alpha\beta}(\mathbf{x},z)dz \tag{4.16}$$

In view of the principle of the virtual work [44], the governing equations in the semi-integral formulation are given as

$$T_{\alpha\beta,\beta}=0\,, \tag{4.17}$$

$$\left(M_{\alpha\beta}-c_1\tilde{M}_{\alpha\beta}\right)_{,\alpha\beta}+c_1\tilde{T}_{3\alpha,\alpha}=-q\,, \tag{4.18}$$

$$c_1\left(\tilde{M}_{\alpha\beta,\beta}-\tilde{T}_{3\alpha}\right)=0 \tag{4.19}$$

and the boundary conditions should obey the following equations at each boundary point

$$n_\beta T_{\alpha\beta}\delta u_\alpha = 0\,, \quad n_\beta \tilde{M}_{\alpha\beta}\delta\varphi_\alpha = 0\,, n_\alpha n_\beta S_{\alpha\beta}\,\delta\left(\frac{\partial w}{\partial \mathbf{n}}\right) = 0\,,\ \left(V - \sum [\![\tilde{T}]\!]\right)\delta w = 0\,.$$

It is appropriate to use dimensionless formulation specified as follows

$$x_\alpha^* = \frac{x_\alpha}{L}\,, \quad z^* = \frac{z}{h_0}\,, \quad u_\alpha^* = \frac{u_\alpha}{h_0}\,, \quad \varphi_\alpha^* = \varphi_\alpha\,, \quad w^* = \frac{w}{h_0}\,, \quad \frac{\partial}{\partial x_\alpha} = \frac{1}{L}\frac{\partial}{\partial x_\alpha^*}\,,$$

$$D_0 := \frac{E_0 h_0^3}{12(1-\nu^2)} \tag{4.20}$$

Then, the governing equations (4.17)-(4.19) result in

$$T^*_{\alpha\beta,\beta} = 0\,, \quad (4.21$$

$$\left(M^*_{\alpha\beta} - c_1\tilde{M}^*_{\alpha\beta}\right)_{,\alpha\beta} + c_1\left(\frac{L}{h_0}\right)^2 \tilde{T}^*_{3\alpha,\alpha} = -q^*\,, \tag{4.22}$$

$$c_1\left(\tilde{M}^*_{\alpha\beta,\beta} - \left(\frac{L}{h_0}\right)^2 \tilde{T}^*_{3\alpha}\right) = 0 \tag{4.23}$$

Let us consider a FGM plate, in which the coordinate dependence of Young's modulus is considered in the factorized form

$$E(\mathbf{x}, z) = E_0 E_H^*(\mathbf{x}) E_V^*(z)\,, \tag{4.24}$$

with assuming the power-law gradation over the thickness of the plate

$$E_V^*(z) = 1 + \zeta\left(\frac{1}{2} \pm \frac{z}{h^*}\right)^p,\ \ \zeta = \frac{E(\pm h^*/2)}{E_0} - 1,\ E_0 = E(\mp h^*/2)\ \ , \tag{4.25}$$

where ζ and p are the values of the level and exponent of the power-law gradation, respectively.

Considering the above mentioned material properties, one can obtain governing equations as

$$d_0^*\left\{D_{1H,\beta}^*\,\tau_{\alpha\beta}^{*(u)}+D_{1H}^*\,\tau_{\alpha\beta,\beta}^{*(u)}\right\}+A_2\,\frac{h_0}{L}\left\{D_{2H,\beta}^*\,\tau_{\alpha\beta}^{*(w)}+D_{2H}^*\,\tau_{\alpha\beta,\beta}^{*(w)}\right\}+$$

$$+A_1\left\{D_{2H,\beta}^*\tau_{\alpha\beta}^{*(\varphi)}+D_{2H}^*\tau_{\alpha\beta,\beta}^{*(\varphi)}\right\}=0 \qquad (4.26)$$

$$\left(d_1^*-c_1A_1\right)\left\{D_{2H,\alpha\beta}^*\,\tau_{\alpha\beta}^{*(u)}+2D_{2H,\beta}^*\,\tau_{\alpha\beta,\alpha}^{*(u)}+D_{2H}^*\,\tau_{\alpha\beta,\alpha\beta}^{*(u)}\right\}+$$

$$+\frac{h_0}{L}\left(A_6-c_1A_{10}\right)\left\{D_{3H,\alpha\beta}^*\,\tau_{\alpha\beta}^{*(w)}+2D_{3H,\beta}^*\,\tau_{\alpha\beta,\alpha}^{*(w)}+D_{3H}^*\,\tau_{\alpha\beta,\alpha\beta}^{*(w)}\right\}+$$

$$+c_1\left(A_5-A_9\right)\left\{D_{3H,\alpha\beta}^*\,\tau_{\alpha\beta}^{*(\varphi)}+2D_{3H,\beta}^*\,\tau_{\alpha\beta,\alpha}^{*(\varphi)}+D_{3H}^*\,\tau_{\alpha\beta,\alpha\beta}^{*(\varphi)}\right\}+$$

$$+c_1\frac{H}{12(1-\nu)}\left(\frac{L}{h_0}\right)^2\left(A_3-c_2A_4\right)\left\{D_{1H,\alpha}^*\left(\frac{h_0}{L}w_{,\alpha}^*+\varphi_\alpha^*\right)+D_{1H}^*\left(\frac{h_0}{L}m^*+\varphi_{\alpha,\alpha}^*\right)\right\}=-\frac{H}{12(1-\nu)}q^* \qquad (4.27)$$

$$A_1\left\{D_{2H,\beta}^*\,\tau_{\alpha\beta}^{*(u)}+D_{2H}^*\,\tau_{\alpha\beta,\beta}^{*(u)}\right\}+A_{10}\,\frac{h_0}{L}\left\{D_{3H,\beta}^*\,\tau_{\alpha\beta}^{*(w)}+D_{3H}^*\,\tau_{\alpha\beta,\beta}^{*(w)}\right\}+$$

$$+A_9\left\{D_{3H,\beta}^*\,\tau_{\alpha\beta}^{*(\varphi)}+D_{3H}^*\,\tau_{\alpha\beta,\beta}^{*(\varphi)}\right\}-$$

$$-c_1\frac{H}{12(1-\nu)}\left(\frac{L}{h_0}\right)^2\left(A_3-c_2A_4\right)D_{1H}^*\left(\frac{h_0}{L}w_{,\alpha}^*+\varphi_\alpha^*\right)=0 \qquad (4.28)$$

In order to decrease the order of derivatives of field variables within the gradients of deformation fields $\tau_{\alpha\beta}^{*(\bullet)}$ in governing equations (4.26)-(4.28), we introduce three new field variables s_α^*, f_α^* and m^* as

$$s_\alpha^*-\nabla^2u_\alpha^*=0\ ,\quad f_\alpha^*-\nabla^2\varphi_\alpha^*=0\ ,\quad m^*-\nabla^2w^*=0 \qquad (4.29)$$

Now, the general system of governing equations given by PDEs with variable coefficients involve not higher than second order derivatives of field variables $\left\{u_\alpha^*,\,\varphi_\alpha^*,\,w^*,\,s_\alpha^*,\,f_\alpha^*,\,m^*\right\}$.

Because of the transversal gradation of Young's modulus, the governing equations (4.26)-(4.28) are not uncoupled for the deflections, rotations and in-plane deformations in FGM plates. Thus, the in-plane deformations are induced in such a FGM plate even if subjected to transversal load only. Furthermore, $T_{33}^{*} \approx T_{\alpha\beta}^{*}$, hence the plane stress conditions are not satisfied either in the generalized sense. In order to investigate the influence of coupling between the deflections and in-plane deformations on the bending stiffness (and so also on the solution of the boundary value problem (BVP)) in the FGM plates with $\zeta \neq 0$, one should compare the results obtained by choosing $c = 0$ and by $c = 1$.

4.2.1 Numerical examples

Owing to strict length limitation, we confine to illustration of just one aspect of the coupling between the bending and in-plane deformation modes due to transversal gradation of Youngs modulus. In presented numerical investigations, we consider a square plate $L \times L$ with clamped all the edges.Poisson's ratio is assumed to be constant $\nu = 0.3$, while various values of the exponent (*p*) and level (ζ) of transversal power-law gradation of Young’s modulus are considered. The transversal uniform static loading $q^{*} = 1$ is applied.

Figure 4.4 illustrates the influence of gradation of Young's modulus on the reduction of deflections against the deflections of homogeneous plate. The results are presented for two levels of gradation (ζ) and various combinations of the c and χ-factors. As expected, the reduction of the deflection is increasing with increasing the level of gradation. The reduction of the maximum deflection is around 20% lower in the case plane stress formulation ($\chi = 1$) than in the case $\chi = 2$ with keeping the same value for the c-factor. This observation is independent on the value of the c-factor. In the case of higher level of gradation, the differences between the results corresponding to $c=0$ and $c=1$ under fixed value of the χ-factor are comparable with differences due to various values of the χ-factor under fixed value of the c-factor. Figure 4.5 shows the in-plane variations of the in-plane displacements, which were generated by $w - u_i$ coupling($c=1$). The influences of ζ and χ-factor are observable.

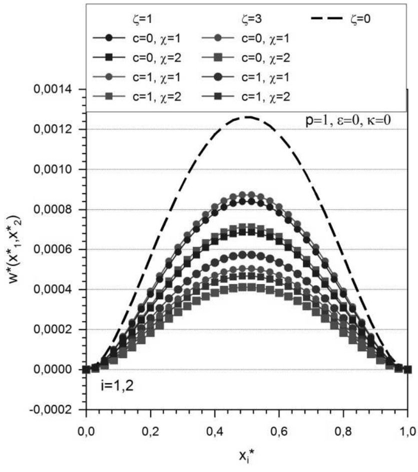

Figure 4.4- In-plane distribution of deflections in transversally graded FGM plates within KLT

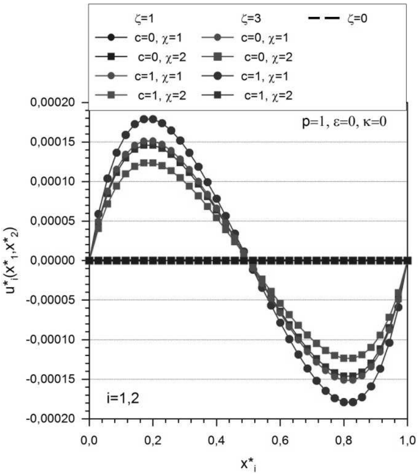

Figure 4.5- In-plane distribution of in-plane displacements in transversally graded FGM plates within KLT

Figures 4.6-4.7 show the dependence of the reduction of maximum value of deflection on the parameter ζ (level of gradation) and the exponent *p* of the power-law gradation, respectively. On Figure 4.6 we can see that the reduction of the maximum value of deflection w.r.t. the $w^*_{ref} = w^*(\zeta = 1, \chi = 1)$ is increasing with increasing the level of gradation under keeping the volume contents of the constituents to be constant ($p = \text{const}$).Figure 4.7 shows the decreasing reduction of the maximum deflection (due to the transversal gradation of Young's modulus) with increasing the value of the exponent of power-law gradation of Young's modulus. This can be explained by the fact that the volume content of the constituent with higher Young's modulus is decreasing with increasing the exponent p .

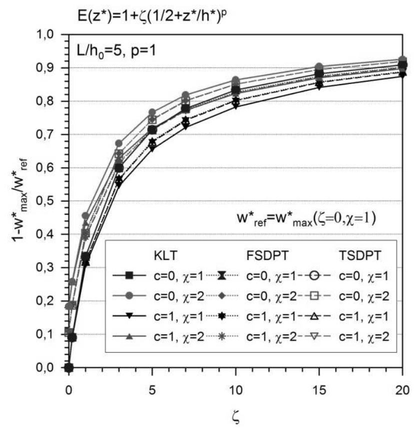

Figure 4.6 Dependence of the reduction of maximal deflection on the level of gradation ζ

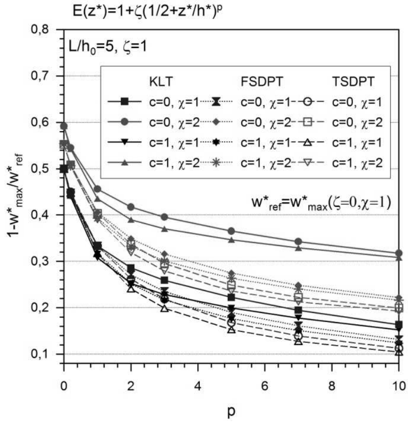

Figure 4.7- Dependence of the reduction of maximal deflection on the exponent of power-law gradation p

5 Conclusions

(i) *One of the main goals of this work is to give a unified derivation of the mathematical model for bending of elastic plates within the generalized theory of thermoelasticty.*

Starting from the energy variational principle in 3D elastostatics (principle of virtual work) and making use of certain geometrical assumptions for plate structures, we could perform the integration w.r.t. the transversal coordinate in closed form and deduce the governing equations as well as the boundary conditions for 2D problems applicable to description of bending and in-plane deformations of elastic plates subject to static loading. **The derived mathematical model involves two key coefficients which can be used for switching among three commonly used theories for bending of elastic plates, such as the Kirchhoff-Love theory (KLT) for bending of thin elastic plates; the shear deformation plate theory of the 1st order (FSDPT) and the SDPT of the 3rd order (TSDPT).**

Allowing transient dynamic loadings and replacing the principle of virtual work by the Hamilton principle, **we derived the mathematical model for 2D problems of vibrations of elastic plates considered within the three theories (KLT, FSDPT, TSDPT).**

Finally, **incorporating the interaction between the elastic and thermal fields within the theory of generalized thermoelasticity, we derived the mathematical model for 2D problems describing transient behaviour of bending modes, in-plane deformations and heat conduction problems in the mid-plane of the plate. Again, one can switch among three bending modes descriptions based on the assumptions used in the KLT, FSDPT and TSDPT.**

In all the derivations, we supposed continuous variation of material coefficients (FGM plates) and variable plate thickness. The variation of material coefficients in the transversal direction must be specified in order to perform the integrations w.r.t. the transversal coordinate in closed form. For this purpose, we assumed the power-law gradation of material coefficients with using two parameters for gradation, such as the level of gradation and the exponent of gradation. As regards the in-plane variations of the material coefficients and the plate thickness, there are no restrictions except the requirement of differentiability.

The transversal gradation of the Young's modulus brings a new physical phenomenon in bending analysis of plates in all three considered theories, namely the coupling between the bending and in-plane deformation modes (between deflections and in-plane displacements) even in plates subject only to transversal loads. Consequently, the plane stress assumptions are not justified even in generalized sense and the plane stress formulation is questionable. The deviations between the results by the plane stress formulation and the formulation without the plane stress assumption can achieve around 20% when the Poisson's ratio is $\nu = 0.3$.

Consideration of functional gradation not only the Young's modulus but also the other material coefficients (such as the mass density, thermal expansion coefficient, heat conduction coefficient, specific heat) **brings a wide variety of effects playing important role in the behaviour of complex coupled system of physical fields**.

(ii) *Another of the main goals of this work is to develop an advanced numerical computational method for solution of initial-boundary value problems resulting from the mathematical model for bending of elastic plates and beams*

The derived governing equations are given by a system of the coupled 4th order PDE with variable coefficients. Thus, one could hardly expect analytical solutions. Therefore advanced numerical computational methods are desired. In any numerical method, the accuracy of approximations of derivatives of field variables is decreasing with increasing the order of derivatives. For this purpose, **we decomposed the formulation with high order derivatives into the formulation for coupled fields with derivatives not higher than 2nd order. The price which should be paid for such decrease of the order of derivatives is introduction of new field variables.**

Getting rid of high order derivatives, **we proposed and developed the strong formulation for solution of complex multifield problems** (with enlarged number of field variables) **represented by systems of the PDE with variable coefficients. The strong formulation** offers the method which **is mathematically as simple as possible** (elimination of any integrations) **and physically correct because of preserving the physical nature of the problems** by recasting the physical principles into the derived mathematical models. **For spatial approximations of field variables, we employed meshless approximations** (such as the MLS approximation and/or the Point Interpolation Method) **which simplify the pre-processing effort** (mesh generation is avoided) **and eliminate certain disadvantages of the mesh based discretization techniques**. Two techniques have been developed for approximation of higher order derivatives of field variables as well as two techniques for evaluation of shape functions. **The mathematical simplification of the computational approach enables increasing the amount of discrete degrees of freedom** (by increasing the amount of nodes) **in order to get efficient and reliable computational method yielding reasonably accurate and numerically stable results.**

The proposed computational technique is quite universal and applicable to solution of boundary value problems for arbitrary systems of PDE **what can be utilized in developing and extending the computer codes for more complex multifield problems.**

(iii) *Finally, the third of the main goals of this work is to test the developed computational method and perform numerical simulations to study the multifield effects in FGM beams and plates*

The accuracy, convergence and numerical stability of the developed computational method have been tested in numerical examples for which exact solutions are available. The computational efficiency has been assessed by taking into account the accuracy and CPU time in comparisons of various variants of the method (e.g. two kinds of evaluation of derivatives of field variables; two kinds of constructing shape functions, …) as well as in comparisons with the weak formulation, where the integration is time consuming handicap. The particular results are discussed mainly in Chapters 4.4, 5.2 and 6.3 of the thesis. **The detailed study of coupling effects has been carried out via numerical experiments in FGM plates** (and beams) **with the transversal as well as in-plane gradation of material coefficients and thickness of the plate** (beam)**.** Interesting conclusions have been drawn when commenting the numerical results in Chapters 4.4, 6.3, 7.2, and 8.4. **It can be concluded that correct treatment of functional gradation of material coefficients in mathematical models is very important because coupling effects arise due to multifield interactions which are missing in plates without gradation of material coefficients.**

Hopefully, the ***developed computer codes bring an efficient computational tool*** for engineers-designers who need to identify and specify the design parameters of construction for an optimal design of structures.

Publications arising from the research

During the research throughout the authors PhD studies, a number of papers have been published based on the work presented in this thesis, which was cited nineteen times by several authors They are listed here for reference.

Publications in international journals registered in Current Contents

SLÁDEK, Vladimír - SLÁDEK, Ján - SÁTOR, Ladislav. *Physical decomposition of thin plate bending problems and their solution by mesh-free methods.* In Engineering Analysis with Boudary Elements, 2013, vol. 37, p. 348–365 (1.596 - IF2012). (2013 - Current Contents). ISSN 0955-7997.

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