This paper presents the boundary element method (BEM) that is applied to the Earth’s gravity field modelling in the Himalayas. A combination of gravimetric and satellite measurements provides the Neumann boundary conditions in the form of surface gravity disturbances that correspond to derivatives of the unknown disturbing potential. This idea has an obvious practical advantage in such mountainous area. An application of BEM to the Neumann geodetic boundary value problem gives a variational (weak) solution directly on the Earth’s surface, where a classical solution could hardly be found. The presented numerical experiments deal with a local refinement of the solution. In order to get accuracy as high as possible, computing on parallel computers is necessary.

1 INTRODUCTION

The determination of an external gravity field is usually formulated in terms of the geodetic boundary value problem (BVP) for the Laplace equation. Our proposed approach is based on the idea that terrestrial gravimetric measurements followed by GPS geocentric positioning are sufficient for a determination of surface gravity disturbances. They almost correspond to derivatives of the unknown disturbing potential at points on the Earth’s surface. This means they represent the Neumann boundary conditions (BC). This fact indicates new possibilities for the gravity field modelling. Although the Neumann geodetic boundary value problem (NGBVP) has theoretically been known for a long time, its practical solution is still an open problem. An application of BEM to NGBVP provides a variational (weak) solution directly on the Earth’s surface, where a classical solution could hardly be found. BEM as a numerical method based on the weak (integral) formulation of the partial differential equation (PDE) is suitable for solving exterior BVPs. The advantage of BEM arises from the fact that only the boundary of the solution’s domain requires a subdivision into its elements. Thus, the dimension of the problem is effectively reduced by one. The formulation of PDE consists of the surface integral equations defined on the boundary, i.e., on the Earth’s surface. They are transformed into a linear system of equations by an appropriate numerical technique, e.g., by the collocation method. Here we consider the collocation technique with linear basis functions. With respect to the giant size of the Earth and in order to get the level of accuracy as high as possible, computing on parallel computers is practically essential. In our applications the number of equations is more than $10^5$. The BEM application enables detailed modelling in a selected region by a local refinement of the triangulation. It gives more precise results inside this area. The presented experiments deal with the numerical aspects of the local refinement in an extremely mountainous terrain – in the Himalayas.
2 THE NEUMANN GEODETIC BOUNDARY VALUE PROBLEM

The disturbing potential \( T \) is defined as the difference between the actual and normal gravity potential. The normal gravity potential is generated by an appropriate normal body, e.g. the geocentric equipotential ellipsoid. Let us assume that the parameters of a normal body and its normal potential are defined in such a way (e.g., the Geodetic Reference System GRS-80 (Moritz, 1992)), that the disturbing potential is a harmonic function outside the Earth. Then it satisfies the Laplace equation

\[
\Delta T(x) = 0, \quad x \in \mathbb{R}^3 - \Omega,
\]

where the domain \( \Omega \) represents the body of the Earth. The surface gravity disturbance \( \delta g \) compares the magnitudes of the actual gravity \( g \) and the normal gravity \( \gamma \) at the same point \( x \)

\[
\delta g(x) = g(x) - \gamma(x), \quad x \in \mathbb{R}^3.
\]

Neglecting a small angle between the directions of \( \gamma \) and the outer normal \( n_e \) to the geocentric equipotential ellipsoid, we can formulate the Neumann BC in the form

\[
\langle \nabla T(x), n_e(x) \rangle \equiv -\delta g(x), \quad x \in \Gamma.
\]

The oblique derivative BC (3) is incorporated into the BEM formulation (4) in the following way. Since the directions of \( V\Gamma(x) \) and \( n_e(x) \) are almost identical, we can approximate the term \( \langle V\Gamma(x), n_e(x) \rangle \) by a projection of the input surface gravity disturbances \( \delta g(x) \) to the vector \( n_e(x) \), i.e. by \( \delta g(x) \cos \alpha(x) \), where \( \alpha(x) \) is the angle \( \angle (n_e(x), n(x)) \). Thus, we can replace the term \( \nabla T(x) \) by \( \delta g(x) / \cos \alpha(x) \) on a triangle. The irregular integrals (singular elements) require special treatment in order to handle the singularity of the kernel function. The integrals with regular integrands, which represent non-diagonal coefficients, are approximated by the Gaussian quadrature rules defined on a triangle. The irregular integrals (singular elements) require special treatment in order to handle the singularity of the kernel function. The details of the numerical procedure are published in (Čunderlík, 2004) and (Čunderlík et al., 2004). Through solving this linear system of equations, we obtain the values of the unknown disturbing potential at the collocation points. Afterwards, they are transformed into height anomalies (quasigeoidal heights) using the modified Bruns formula in an iterative way, which is symbolically written using the expression

\[
\zeta^{i+1}(\phi, \lambda) = \frac{T(\phi, \lambda, h)}{\gamma(\phi, h - \zeta^i)},
\]

where \( \phi, \lambda, h \) are the ellipsoidal (geodetic) coordinates of the collocation point; \( \zeta^{i}(\phi, \lambda) \) is the height anomaly of the \( i \)-th iteration; \( T(\phi, \lambda, h) \) is the disturbing potential at the collocation point; and \( \gamma(\phi, h - \z) \) is the normal gravity on the “iterative” telluroid.

3 BEM APPLIED TO NGBVP

In a direct BEM formulation the boundary integral equation is derived from the weak (integral) formulation of the Laplace equation (1) through the application of Green’s second theorem (Brebbia et al., 1984). In three dimensions it has the following form

\[
4\pi T(x) + \int T(y) \frac{\partial G}{\partial n_e}(x, y) \, dy = \int \frac{\partial}{\partial n_e}(y) G(x, y) \, dy, \quad x \in \Gamma,
\]

where \( n_e \) is the normal to the boundary \( \Gamma \). The kernel function \( G \) is known as Green’s function, and it represents the fundamental solution of the Laplace equation

\[
G(x, y) = \frac{1}{4\pi|x - y|}, \quad x, y \in \mathbb{R}^3.
\]
memory requirements are the main limitation factor for any computations. Thanks to the opportunity to access the HPC facilities (see the acknowledgments) it was possible to compute on several processors and consequently to extend our computations. The code parallelization was performed by the standard MPI subroutines (Aoyama, Nakano, 1999). Point-to-point and collective communication subroutines were implemented for the matrix assembly as well as for the linear solver BiCGSTAB (Barrett et al., 1994).

Moreover, the elimination of interactions of the far zones can rapidly reduce the memory requirements. This can be performed straightforward using the fast multipole methods (Greengard, Rokhlin, 1987) or simply incorporating the known geopotential models. Both methods can transform the original dense system matrix into a sparse one. Here we consider the second approach, i.e., we use the low-frequency components of the geopotential models accurately obtained by satellite measurements.

5 NUMERICAL EXPERIMENTS

The numerical experiments deal with the gravity field modeling in an extremely mountainous terrain – the Himalayas (Fig.1). The BEM applications enable local refinements of the solution in the selected region. First, the Earth's surface is approximated by an approximately homogenous global triangulation of the topography (Čunderlík et al., 2004). Then, the global triangulation is sequentially refined in the selected region until the required level (Fig.2). In such a way we can generate the horizontal positions of the collocation points. The vertical components, i.e., the ellipsoidal heights, are interpolated from a combination of the known geopotential model EIGEN-CG01C (GFZ Potsdam) and the global digital elevation model GTOPO-30 (EROS Data Centre). Unfortunately, the original gravity data in this area are not available. Therefore, the surface gravity disturbances at the collocation points are generated from the geopotential coefficients of EIGEN-CG01C (Fig.3). Consequently, the variational solution of our application should converge on the EIGEN-CG01C
geopotential model. Such an experiment should confirm the mathematical reliability of the proposed solution and estimate its accuracy.

The large-scale computations were accomplished using the high-speed parallel computer IBM SP4 with 512 processors (1.2 TB of the distributed internal memory) at CINECA (see the acknowledgements). In order to extend the computations as much as possible, the interactions of the far zones were eliminated by incorporating the known geopotential models (Tab. 1). Here, 3 experiments were considered:

a) the homogenous global triangulation for $\Delta \phi = 0.5^\circ$ (194 402 collocation points) with the elimination factor $a/2$ (an effect of the far zones’ interaction further than $a/2$ are considered known, where $a$ is the semi-major axis of the Earth)

b) the local refinement of a) in the Himalayas (Fig. 2) by 2 refined “halving” levels, i.e., $\Delta \phi = 0.125^\circ$ in fully refined zones (247 273 collocation points) with the elimination factor $a/10$

c) homogenous global triangulation for $\Delta \phi = 0.36^\circ$ (375 002 collocation points) with the elimination factor $a/8$

The variational solution as a result of the BEM applied to NGBVP is compared with the EIGEN-CG01C geopotential model. Figure 4 depicts the local quasigeoid model in the Himalayas (experiment b)). Its residuals from EIGEN-CG01C are shown in Fig. 5. The statistical characteristics of the residuals in the selected region as well as the computational aspects are depicted in Tab. 1.

6 ANALYSES OF THE RESULTS

The numerical results of the proposed approach show a strong mathematical reliability. The comparison between the variational solution and the EIGEN-CG01C geopotential model indicates the high accuracy of the achieved results, considering such a mountainous terrain. The local refinement has resulted in a more precise solution (compare experiments a) and b) in Tab. 1). According
to the statistical characteristics in Tab.1, a further refinement of the global triangulation has surprisingly resulted in the best solution (experiment c)). However, the depicted residuals show a better overall agreement of the local refinement (experiment b)) except for small zones of the high extremes. It could be due to a fact that in our approach the input data as well as the variational solution are determined on the Earth’s surface. In the case of the geopotential models (based on the spherical harmonics) the boundary is a sphere or ellipsoid and does not reflect the complicated mountainous terrain. This could describe the reason for the higher residuals in experiment b) than in c).

The depicted residuals (Fig.5) show that the main disagreements are on the edges of mountain ranges where the quasigeoid is extremely undulated (zones of the highest vertical of the deflection). Here, a small inaccuracy in horizontal directions results in striking residuals in the vertical directions. A refined discretization of the extremely mountainous terrain allows modelling of detailed quasigeoid undulations in the areas of steep peaks (Fig.5b). The overall depression of the variational solution in comparison with the EIGEN-CG01C confirms the fact that the low-frequency components of the geopotential models accurately obtained by satellite measurements cannot be improved by terrestrial gravimetric data (Klees et al., 2001).

7 CONCLUSIONS

The boundary element method applied to the Neumann geodetic boundary value problem provides the variational solution directly on the Earth’s surface. The numerical experiments show the obvious perspective of the proposed approach for the gravity field modeling in an extremely mountainous terrain – the Himalayas. A local refinement provides a more precise solution inside the refined region. The elimination of far zones’ interactions reduces the memory requirements rapidly. This permits further computation extensions. The Neumann boundary conditions in the form of the surface gravity disturbances have a striking practical advantage as the gravimetric measurements can be followed by precise GPS positioning without demanding levelling.

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