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THE USE OF A KALMAN FILTER IN GEODESY AND NAVIGATION

ABSTRACT
A Kalman filter is an optimal linear estimator developed in 1960. It is a recursive algorithm, i.e. it uses its output as an input for the next cycle. It allows to effectively estimate the dynamic parameters and predict their future values.

Although huge number of papers have been written on this topic, this paper intends to introduce the processes within a Kalman filter and allow them to be fully understood without the need for a broad mathematical background. Several examples of a Kalman filter based on situations in geodesy, surveying and navigation are solved step by step to explicitly show the performance of this approach to linear estimations.

1 PREFACE
In the past, the determination of angles, lengths or heights was considered as an observation in land surveying or geodesy. These observations required much time and could therefore not be repeated very often. The amount of data for further processing was quite low. In the first half of the 19th century the Least Square Method (LSM) [Gauss, 1823] was developed. The procedure provides an estimate of certain parameter(s) out of a number of observations. This method minimizes the sum of the squares of differences between the observation and estimate.

Thanks to the development of technology there are huge sets of data to be processed nowadays. Especially since the Global Navigation Satellite Systems (GNSS) appeared and were developed, the interval of repeatedly observed position has continually been getting shorter from days to hours, minutes and even seconds and fractions of seconds. Such enormous amounts of data are not possible to effectively process using the plain LSM, as the processing requires large matrix operations (inversions) that slow down the whole process. This problem can easily be resolved by using a recursive algorithm utilizing part of its output as an input for the next iteration. One of these algorithms based on LSM is the Kalman Filter (KF) [Kalman, 1960]. Several of its simple modifications and examples are described in this paper.

The paper consists of two main parts. The first one provides the theory of KF, while the second part demonstrates the use of KF with practical/illustrative examples.

Variables
There are several conventions for signing the variables used in the paper:
- $a$ – scalar variable (a number),
- $a$ – vector,
- $A$ – matrix,
- $A^T$ – transposed matrix $A$,
- lower index $k$ indicates the epoch $k$,
- $\hat{X}$ is the first and $\hat{X}'$ is the second derivation of $X$, $\hat{X}'$ means the estimate,
sometimes the upper index $^+$ appears to emphasize dealing with an 
a posteriori estimate, upper index $^-$ means the Kalman filter a priori 
estimate just before the measurement [Gelb 1974]. $I$ represents the 
identity matrix and $t_k$ indicates the epoch $k$ of the observation.

2 THE PRICIPLES OF THE KALMAN FILTER

Assume that $z$ consists of $l$ measurements that can be expressed as 
a linear combination of $n$ elements of estimate $x$ and measurement 
error $v$.

$$
z = f(x) + v = Hx + v,$$

where $H$ is a design matrix consisting of partial derivations of 
measurements [Mervart, 1994]

$$
H = \frac{\partial f(x)}{\partial x^T}.
$$

The Kalman filter is an optimal recursive algorithm for data processing 
[Maybeck, 1979]. Assuming a correct model of the measurement and 
observed process, only the current epoch observation and the last 
estimate have to be used. The algorithm is a tool that estimates the 
current state of the dynamic system out of incomplete noisy indirect 
measurements. However, it can also be used when resolving static 
problems, or using complete or direct measurements. KF is suitable 
for linear as well as nonlinear processes. The principle of KF is based on 
two basic phases of the process: prediction and update.

Prediction

Prediction predicts the state of the system (i.e. the estimated value) 
and its covariance matrix in epoch $k$ based on its estimate for the 
preceding epoch $k-1$.

$$
\hat{x}_k = \Phi_k \hat{x}_{k-1} + B_k u_k,
$$

$$
P_k^- = \Phi_k P_{k-1}^- \Phi_k^T + B_k \Gamma_k B_k^T + Q,
$$

where $\hat{x}$ represents the estimated parameter, $P$ its covariance 
matrix, $B$ is the control matrix used for binding the known 
parameters $u$ with the estimated parameters, $\Gamma$ is the covariance 
matrix of the known input values $u$, $\Phi$ is the transformation matrix 
defining the relationship between the preceding and current epoch, 
$Q$ is the white noise covariance matrix of the process. The influence 
of the noise matrix is described in section 3.7.

Usually, there are no other known values in land survey and geodesy; 
therefore, Eqs. (3) and (4) can be reduced to a simplified form

$$
\hat{x}_k^- = \Phi_k \hat{x}_{k-1}^-,
$$

$$
P_k^- = \Phi_k P_{k-1}^- \Phi_k^T + Q
$$

These modified equations are sufficient for most geodetic 
applications. The Kalman filter in its general form is discussed in 
section 3.5.

Update

Kalman gain matrix $K_k$ represents the weight of the actual 
measurement with respect to the estimated value.

$$
K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1}
$$

where $R_k$ is the observation covariance matrix. The basic analysis 
of Eq. (7) shows that a more precise measurement (i.e., the lower 
covariance matrix elements) raises its weight [Welch-Bishop, 2001]

$$
\lim_{R_k \to 0} K_k = H_k^{-1}.
$$

This form is very illustrative, however, as $H_k$ is generally a non-
square matrix and thus cannot be inverted, Eq. (8) should be stated 
in the following form

$$
\lim_{R_k \to 0} K_k H_k = I.
$$

On the other hand, the a priori covariance matrix approaching zero 
values means the low weight of the observation and a priori residual 
[Welch-Bishop, 2001]

$$
\lim_{R_k \to 0} K_k = 0.
$$

Suppose the apriori residual $e_k$ as the difference between the current 
observation and the expected observation determined in the last 
parameter estimate

$$
e_k = z_k - H_k \hat{x}_k^-.
$$

Then the updated estimate can be determined as

$$
\hat{x}_k^+ = \hat{x}_k^- + K_k e_k
$$

and its updated covariance matrix

$$
P_k^+ = (I - K_k H_k) P_k^-.
$$

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Initialization

The aforementioned process uses the preceding estimate; therefore, it requires the initial values, just like every recursive algorithm. It is a simple process defining the basic values for epoch 0.

Vector $\mathbf{x}_0$ can be either assigned with an initial measurement or it can be defined as zero. When initializing a covariance matrix, the most suitable approach is to define the diagonal matrix with large elements that provide almost zero weight of the a priori estimate in the next iteration. The performance of the covariance matrix is described in section 3.2.

3 THE EXAMPLES

In this section we apply the knowledge gained from the previous chapter to several practical examples from simple to more complex applications. All the examples use a discrete linear form of the Kalman Filter (KF). The Extended Kalman Filter (EKF) for continuous measurements is described in detail, for example, in [Gelb 1974].

3.1 One-dimensional stationary parameter

The most simple example of KF is an estimate of a one-dimensional stationary variable out of a set of direct measurements. A stationary parameter means that it does not change over time. An example of such a process can be the repeated measurement of a distance or level height.

Assume $h$ is the estimated level height, $z$ – an observed level height. The initial value is $h_0 = 0$ or, even more suitably in this case, $h_0 = z_1$. The covariance matrix is replaced by variance when using a one-dimensional parameter. As the initial RMS of the estimated parameter is unknown, a large enough number should be used to ensure the initial estimate will not influence any further cycles significantly. One (1) should be appropriate for this purpose (i.e. the uncertainty of a levelled height is 1 meter). The white noise of the process is estimated empirically. Some sources (e.g. Welch-Bishop, 2001) encourage looking for the most appropriate noise value(s) empirically.

In this example the noise of the levelling measurements is $w = (0.1 \text{ mm})^2$. As there is only one measurement for each observation in this case, RMS cannot be determined. Suppose the RMS for one measurement is 1 mm, i.e. $r = 1 \text{ mm}^2 = 10^{-6} \text{ m}^2$. The parameter to be estimated is a time invariant; therefore, the transformation matrix does not apply, i.e. it is replaced by an identity matrix (in the case of one-dimensional parameter, the transformation coefficient is set to 1). In this example the estimated parameter is directly observed; therefore, the design matrix $\mathbf{H}$, expressing the relationship between the observation and estimated parameter, is also identity ($\mathbf{H} = \mathbf{I}$, in the case of 1-D parameter $H = 1$).

After applying these rules to (5) and (6), it can be stated that

\[
\begin{align*}
\hat{h}_1 &= \hat{h}_0 = 0 \\
\mathbf{P}_1 &= \mathbf{P}_0 + \mathbf{w} = 1 + 10^{-6}.
\end{align*}
\]

After observing the height, the estimates can be updated according to equations (7)-(10):

Kalman gain $k_1 = \frac{\mathbf{P}_0 (\mathbf{H}^\mathbf{T} \mathbf{I} + \mathbf{R})^{-1}}{\mathbf{10000001}} \approx 0.999999$

A priori residual $\mathbf{\hat{e}}_1 = z_1 - \hat{h}_1 = z_1$

Updated height estimate $\hat{h}_1 = \hat{h}_1 + k_1 \mathbf{\hat{e}}_1 \approx 0.999999 z_1$

Updated RMS estimate $\mathbf{P}_1 = (1 - k_1) \mathbf{P}_0 \approx 0.000001 \mathbf{P}_0 \approx 10^{-4}$

It has been shown above that already in the first cycle of the Kalman filter, the estimate has come close to the observed value of the levelled height.

The parameters $\hat{h}_1$ and $p_1$ are the output values of the first iteration of the Kalman filter, and they also serve as the input values for the second iteration.

3.2 Multidimensional stationary parameter

The next section will demonstrate the use of a Kalman filter for estimating a multidimensional parameter. As an example, we use determination of the 3D position in a local coordinate system $\mathbf{neu}$. It is actually the same case as described in 3.1. This example has been chosen to show the performance of matrices in a Kalman filter. Suppose the repeatedly measured 3D vector $z = [n, e, u]^T$ in a local coordinate system. The goal is to regularly update its estimated value.

The initialization will be carried out similarly to the first case: the estimated vector will be zero and the covariance matrix will be identity (the diagonal elements can be arbitrary – the larger the number, the less weight the initial estimate has).

\[
\begin{align*}
\mathbf{x}_0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \mathbf{P}_0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]

As in this case we are dealing with a stationary parameter, the transformation matrix shall be identity $\mathbf{\Phi} = \mathbf{I}$. For this example suppose that the white noise value is 1 mm ($10^{-6}$) for the horizontal elements and 3 mm ($9 \cdot 10^{-6}$) for the vertical element.

The estimated parameter is observed directly (actually it is observed indirectly via GPS or similar means but we receive the final result of the observation in the form of a coordinate triplet – either in...
a geocentric coordinate system or transformed to a local system and its covariance matrix); therefore, $H = I$. The prediction for the first epoch shall be carried out as follows:

$$\hat{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad P_0 = P_0 + Q = \begin{bmatrix} 1.000001 & 0 & 0 \\ 0 & 1.000001 & 0 \\ 0 & 0 & 1.000009 \end{bmatrix}.$$

For the first epoch suppose RMS of the observed parameters $n$, $e$, $u$ to be 1, 2 and 5 mm (e.g. the covariance matrix diagonal components are $1.10^{-6}$, $4.10^{-6}$ and $2.5 \cdot 10^{-5}$). The parameter estimate will proceed by substituting for equations (7)-(12):

- Kalman gain

$$K = P_0 + R = \begin{bmatrix} 1.000001 & 0 & 0 \\ 0 & 1.000001 & 0 \\ 0 & 0 & 1.000005 \end{bmatrix}.$$

- a priori residual

$$\tilde{e}_0 = z_0 - \hat{z}_0 = \begin{bmatrix} n_0 \\ e_0 \\ u_0 \end{bmatrix}.$$

- updated estimate of position vector

$$\hat{x}_0 = K \tilde{e}_0 = \begin{bmatrix} 1.000001^{-1} & 0 & 0 \\ 0 & 1.000004^{-1} & 0 \\ 0 & 0 & 1.000025^{-1} \end{bmatrix} \begin{bmatrix} n_0 \\ e_0 \\ u_0 \end{bmatrix}.$$

- updated covariance matrix

$$P_0 = (I - K) P_0 = \begin{bmatrix} 0.000001 & 0 & 0 \\ 0 & 0.000004 & 0 \\ 0 & 0 & 0.000025 \end{bmatrix}.$$

### 3.3 Dynamic problems

A change in position does not necessarily mean an error. In the case the measuring device is moving, the observed position changes. It is suitable to include the velocity estimate and/or direct observation of the velocity elements or its absolute value into the process modeling when solving dynamic problems.

#### 3.3.1 Direct measurement

Suppose a vehicle with a GNSS receiver monitoring its current position. The observed parameter is the geocentric position of the vehicle and its change over time (velocity) $z_t = [\hat{x}, \dot{\hat{x}}, \ddot{\hat{x}}, \dot{\hat{y}}, \ddot{\hat{y}}, \dot{\hat{z}}, \ddot{\hat{z}}]$ with a corresponding covariance matrix $R_t$; the estimated parameter is also the geocentric position and the velocity $\hat{z}_t = [\hat{x}, \dot{\hat{x}}, \ddot{\hat{x}}, \dot{\hat{y}}, \ddot{\hat{y}}, \dot{\hat{z}}, \ddot{\hat{z}}]$ and its covariance matrix $P_t$. It is obvious that the estimated parameter is a directly observed value; therefore, the design matrix is identity. This means that the real value is estimated out of the noisy measurements of the same parameter. The transformation matrix providing the parameter estimate based on the previous estimate is defined as

$$P_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By substituting for (3) we get

$$\hat{x}_t = \hat{x}_{t-1} + \dot{\hat{x}}_{t-1}(t_t - t_{t-1}) + \ddot{\hat{x}}_{t-1}(t_t - t_{t-1})^2/2,$$

The predicted covariance matrix $P_t$ can be determined similarly using Eq. (6).

In the event the acceleration is known, the model can be easily adjusted for using this value. To simplify the equation form we adopt the substitution $\mathbf{r}_t = [\hat{x}, \dot{\hat{x}}, \ddot{\hat{x}}, \ddot{\hat{z}}]$. The Eq. (5) will thus take the form

$$\hat{r}_t = \Phi \hat{r}_{t-1} = \begin{bmatrix} 1 & \Delta t & \Delta t^2/2 & 0 \\ 0 & 1 & \Delta t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_{t-1} \\ \dot{\hat{x}}_{t-1} \\ \ddot{\hat{x}}_{t-1} \\ \ddot{\hat{z}}_{t-1} \end{bmatrix}.$$

Again, one of the parameters, in this case, the acceleration, stays unchanged, while the other values vary according the previous estimate.

#### 3.3.2 Number of estimated parameters exceeding the observed values

Contrary to the classic LSM, the Kalman filter does not require – by definition – the number of observed parameters in one epoch to be
higher or equal to the number of estimated parameters. Stationary and quasi-stationary processes (i.e., processes that change in time while parameters describing them stay unchanged) allow us to determine the estimated parameters gradually with an increasing number of observations. However, the total number of observations must be higher than the number of estimated parameters. At the beginning all the estimated parameters should be initialized; in the next steps the estimated parameters are being gradually determined. In the next example we will see how the Kalman filter works if the number of estimated parameters exceeds the number of observations in one epoch. To keep the example simple, suppose one-dimensional observation and two-dimensional estimate.

Suppose an object moving constantly in one direction (line). Its position \(d\) is being determined in regular intervals with RMS \(r = 0.1\). This object can be, for instance, a buoy representing the depth of a water source while being constantly filled. The goal is to determine the position \(d\) and the velocity \(v\) in epoch \(k\).

The input values for the next example have been generated with a starting point \(d_0 = 20\) and velocity \(v = 2\) altered by a random error added with a normal distribution \(\epsilon \sim N(0,0.1)\). The observed positions in various epochs are \(z = [20.039, 21.898, 24.101, 26.055, ...]^T\).

Assume \(x = [d, v]^T\). The initialization process provides the initial estimates and the weight matrix:

\[
\hat{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad P_x = \begin{bmatrix} 1000 & 0 \\ 0 & 1000 \end{bmatrix}
\]

As we suppose a constant velocity, the following relationships between epoch \(k\) and \(k-1\) can be stated:

\[
\begin{align*}
\hat{d}_k &= \hat{d}_{k-1} + \hat{v}_{k-1} \Delta t \\
\hat{v}_k &= \hat{v}_{k-1}
\end{align*}
\]

The design matrix and the transformation matrix thus take the following forms:

\[
\Phi = \frac{\partial x_{k+1}}{\partial x} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}, \quad H = \frac{\partial z}{\partial x} = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

Assume the measurements are observed in a constant interval \(\Delta t = 1\) and

\[
Q = \begin{bmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{bmatrix}
\]

Following the Eqs (7)-(12) we determine the Kalman gain \(K\) and a priori residual \(\hat{v}\)

\[
K = \begin{bmatrix} 0.999995 \\ 0.499997 \end{bmatrix}, \quad \hat{v} = 20.039
\]

The first iteration provides the position and velocity estimates and the corresponding covariance matrix:

\[
\hat{x}_1 = \begin{bmatrix} 20.039 \\ 10.0194 \end{bmatrix}, \quad P_x = \begin{bmatrix} 0.1 & 0.005 \\ 0.005 & 500.003 \end{bmatrix}
\]

A closer inspection of the covariance matrix reveals that the first cycle of the Kalman filter provides no real value for the velocity estimate. The position estimate, on the other hand, is equal to the observed value while its RMS is quite low already. A better estimate for the velocity will appear in the next cycles of the Kalman filter:

\[
\hat{x}_2 = \begin{bmatrix} -8.16 \\ 21.898 \end{bmatrix}, \quad P_x = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}
\]

\[
\hat{x}_3 = \begin{bmatrix} 0.343 \\ 24.044 \end{bmatrix}, \quad P_x = \begin{bmatrix} 0.008 & 0.005 \\ 0.005 & 0.005 \end{bmatrix}
\]

\[
\hat{x}_4 = \begin{bmatrix} -0.021 \\ 26.061 \end{bmatrix}, \quad P_x = \begin{bmatrix} 0.007 & 0.003 \\ 0.003 & 0.002 \end{bmatrix}
\]

The technique of estimating the parameters exceeding the observed values described above is suitable for stationary processes. The processing of dynamically changing parameters exceeding the input values is described in [Welch – Bishop 1997].

### 3.4 Periodic processes

In some cases it is possible to re-formulate a dynamic problem to a stationary one. Using this method the transformation matrix becomes identity and thus can be eliminated from the process model. An example of such a process can be a periodic one.

There are many observed processes, including permanent GNSS measurements, influenced by periodic phenomena such as night and day alternations, the seasons, etc. To model such processes, it is possible to use a harmonic function such as

\[
z = f(t) = a + b t + c \sin \frac{2 \pi t}{T} + d \cos \frac{2 \pi t}{T},
\]

where \(T\) represents the period of the influencing phenomenon, and \(t\) is the epoch of the observation. The period can be either determined by a priori analysis or we can use basic periods such as one year when modeling the seasonal variation or one day when modeling diurnal processes.

In this case the parameters to be estimated are the curve parameters \(x = [a, b, c, d]^T\), where \(a, b\) are parameters of the trend line and \(c, d\) are the amplitudes of the harmonic functions. Although the
measurement equation itself is nonlinear, the estimated parameters are only the coefficients \( \mathbf{k} \) (in case the period is known). The problem can be solved by the methods described above.

The design matrix will take the following form

\[
\mathbf{H} = \frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix}
1 & t_x & \frac{2\pi t_x}{T} & \cos \frac{2\pi t_x}{T}
\end{bmatrix}
\]

Similarly to the example in section 3.3.2, there are fewer measurements (1) in a single epoch than there are estimated parameters (4). Initial estimates can either be derived from a year-long observation and following analysis or can be defined arbitrarily. When following the first approach, it is also suitable to estimate the weighting coefficients to utilize the preceding analysis output. The second approach, on the other hand, requires the covariance matrix with the diagonal elements as large as possible (so that the weights can be as low as possible).

### 3.5 General form of the Kalman filter

Sometimes there are also known values as being a part of the process besides the observed parameters. In such cases the process requires adding the control input so that we can use the general equations of the Kalman filter (3) and (4). An example of 2-D localization will be used to explain the general form.

Suppose a robot (an automatic lawn mower, for instance), whose position and orientation is expressed by vector \( \mathbf{x} = [a, b, \theta]^T \), where \( a, b \) are 2-D coordinates and \( \theta \) is orientation. The position is determined trilaterally based on radio signals from beamers placed in the test area. The movement is described by control vector \( \mathbf{u}_t = [\Delta \mathbf{D}_1, \Delta \mathbf{D}_2]^T \), where \( \Delta \mathbf{D} \) is the distance traveled by the point at the center of the robot’s front axle, obtained by averaging the distances measured by the left and right wheel encoders. The incremental orientation change \( \Delta \theta \) is obtained by the onboard gyro.

The position and orientation in epoch \( k+1 \) can be described by the following set of nonlinear equations:

\[
g(\mathbf{x}_{k+1}) = \begin{bmatrix}
\alpha_x + \Delta \mathbf{D}_1 \cos(\theta_k) \\
\beta_x + \Delta \mathbf{D}_2 \sin(\theta_k) \\
\theta_x + \Delta \theta_k
\end{bmatrix}
\]

The transformation matrix is constituted of partial derivatives of the process equations by the estimated parameters

\[
\mathbf{F}_{k+1} = \frac{\partial f(\mathbf{x}_{k+1})}{\partial \mathbf{x}} = \begin{bmatrix}
1 & 0 & -\Delta \mathbf{D}_1 \sin(\theta_k) \\
0 & 1 & \Delta \mathbf{D}_2 \cos(\theta_k) \\
0 & 0 & 1
\end{bmatrix}
\]

The control process matrix \( \mathbf{B} \) binds the control and estimated parameters. Its elements are partial derivatives of the process equations by control parameters \( \mathbf{u} \)

\[
\mathbf{B} = \begin{bmatrix}
\cos(\theta_k) & 0 \\
\sin(\theta_k) & 0 \\
0 & 1
\end{bmatrix}
\]

The position of the robot is determined by radio signals from broadcasters placed at positions \([a_v, b_v]^T\). The observed parameter is the distance from the broadcaster. The observation equation is

\[
h(\mathbf{x}_{k+1}, [a_v, b_v]^T) = \sqrt{(a_{x+1} - a_v)^2 + (b_{y+1} - b_v)^2}
\]

and its partial derivations by the estimated parameters constitute the design matrix

\[
\mathbf{H} = \frac{\partial h}{\partial \mathbf{x}} = \begin{bmatrix}
\frac{a_{x+1} - a_v}{\sqrt{(a_{x+1} - a_v)^2 + (b_{y+1} - b_v)^2}} \\
\frac{b_{y+1} - b_v}{\sqrt{(a_{x+1} - a_v)^2 + (b_{y+1} - b_v)^2}}
\end{bmatrix}
\]

A further solution gets more complicated as it requires the determination of white noise covariance matrix \( \mathbf{Q} \). The importance of the matrix is discussed briefly in section 3.7. The general model is described in detail in [Kurth, 2004] from where the robot example has also been adopted.

### 3.6 Extended Kalman filter

The aforementioned examples have either been linear problems or transformed to be linear. In reality, most of the processes cannot be easily described by linear equations; nonlinear functions should be used instead. To deal with such processes a special method has been developed, known as the Extended Kalman Filter.

For the parameter being estimated there is a nonlinear equation

\[
x = f(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k)
\]

where \( \mathbf{x} \) is the estimated parameter vector, \( \mathbf{u} \) is the control input and \( \mathbf{w} \) is the random error vector. The design matrix is replaced by function matrix \( \mathbf{h} \):

\[
\mathbf{z} = \mathbf{h}(\mathbf{x}_k, \mathbf{v}_k)
\]

The equations are further linearized and thus the random error will not be normally distributed anymore. This topic is rather broad and therefore will not be discussed in this paper. The Extended Kalman Filter is discussed in detail in many papers, including [Gelb 1974] or [Welch-Bishop, 2001].
3.7 White noise covariance matrix

The white noise covariance matrix $Q$ represents the effect of a random error on an actual observation. In the event the process is considered to be too noisy, the weight of the current observation gains. And vice versa. If the minimal noise is assumed, the weight of the new observation will be correspondingly low. Therefore, it is very important to find and use noise values suitable for the observed process.

Figure one shows a set of one-dimensional stationary values processed by the Kalman filter. Various levels of a white noise have been applied. The set consists of 100 computer-generated observations with a standard deviation of 0.5. This value has also been used as a constant RMS $r$ for all the measurements. The line dividing the graph into two parts represents the mean value of the random variable.

The output values prove the statements of the preceding sections. If the noise is set to $q = 0.1$ or $q = 0.01$, the processed values actually follow the current observations. However, using the noise level $q = 0.001$ or lower, the Kalman filter outputs a rather smooth curve. A noise level lower than $q = 0.0005$ does not significantly improve the results; therefore, it is suitable to use the noise from interval $<0.001; 0.0005>$ for this case.

When the actual value suddenly changes, the system might not work properly if the low noise level is applied. The lower the noise, the longer the period of time needed to adopt estimates to new situation. Therefore, if there is the possibility of such a sudden change during the observed process, it should be considered in the initial project modeling. It can be handled by various tests, temporarily lowering the noise values, etc. On the other hand, when it is impossible for such a change to appear during the process observation, the Kalman filter reliably eliminates it as an outlying observation.

4 CONCLUSIONS

The Kalman filter is an optimal tool for solving various problems. Its use in the field of geodesy and land surveying is simple and easily modifiable to fit the user’s needs. However, the definition of the examined process might be problematic for people who are specialists in other fields and are not used to complicated mathematical formulations. The goal of this paper is to help specialists in surveying and geodesy to better understand the principle of the Kalman filter just by using the basic knowledge of matrix algebra and to encourage use of this effective method of adjusting measurements. The importance of a suitable noise level setting is also discussed and graphically explained.
REFERENCES


